

Probability Theory 2

5th Exercise Sheet: Concentration inequalities I

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- 5.1** (a) Show that the Markov inequality is **sharp**. Namely, for every fixed real numbers $0 < m \leq \lambda$ there exists a random variable X such that $\mathbb{E}(X) = m$ and

$$\mathbb{P}(X \geq \lambda) = m/\lambda.$$

- (b) Show that the Markov inequality is **not sharp**. Namely, for every fixed non-negative random variable X with finite expected value

$$\lim_{\lambda \rightarrow \infty} \lambda \mathbb{P}(X \geq \lambda) / \mathbb{E}(X) = 0.$$

- 5.2** Let X be a random variable such that $\mathbb{E}(X) = 100$ and $\mathbb{P}(X < 30) = 0$. Give the best estimate for $\mathbb{P}(X > 70)$.

- 5.3** Let $X_1, \dots, X_9$ be independent random variables with distribution $UNI[0, 1]$. Moreover, let $Y = \sqrt[9]{X_1 \cdots X_9}$. Using Chebysev's inequality, give a lower estimate for the probability

$$\mathbb{P}(e^{-5/3} < Y < e^{-1/3})!$$

- 5.4** Let $X_1, X_2, \dots$ be random variables with finite variance, and 0 expected value (i.e. for every $i \geq 1$, $\mathbb{E}(X_i) = 0$, $\sigma_i^2 := \mathbb{D}^2(X_i) = \mathbb{E}(X_i^2) < \infty$). Let r_n be a sequence such that $\lim_{n \rightarrow \infty} r_n = 0$ and suppose that $\mathbf{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) \leq r_{|i-j|}$ for every $i, j \geq 1$ (In particular, $\sigma_i^2 \leq r_0$). Let $S_n := X_1 + X_2 + \dots + X_n$. Show that $\lim_{n \rightarrow \infty} \mathbb{P}(|S_n/n| > \delta) = 0$ for every $\delta > 0$.

- 5.5** Let $X_1, X_2, \dots$ be uncorrelated random variables with finite variance, 0 expected value. (That is, for every $i \geq 1$, $\mathbb{E}(X_i) = 0$, $\sigma_i^2 := \mathbb{D}^2(X_i) = \mathbb{E}(X_i^2) < \infty$, and for every $i \neq j$, $\mathbb{E}(X_i X_j) = 0$). Let $S_n := X_1 + X_2 + \dots + X_n$. Show that if $\lim_{i \rightarrow \infty} \sigma_i^2/i = 0$ then $\lim_{n \rightarrow \infty} \mathbb{P}(|S_n/n| > \delta) = 0$ for every $\delta > 0$.

- HW*** **5.6** (WLLN for renewal processes) Let $\tau_1, \tau_2, \dots, \tau_n, \dots$ be i.i.d non-negative random variables. Suppose that $\mathbb{E}(\tau_i) =: m < \infty$. Let $T_n := \sum_{i=1}^n \tau_i$ and let $\nu_t := \max\{n : T_n \leq t\}$. The WLLN states that for every $\delta > 0$, $\lim_{n \rightarrow \infty} \mathbb{P}(|T_n/n - m| > \delta) = 0$. Prove that the dual statement is also true, namely, for every $\delta > 0$

$$\lim_{t \rightarrow \infty} \mathbb{P}(|\frac{\nu_t}{t} - m^{-1}| > \delta) = 0.$$

Hint: Figure out what $|\frac{\nu_t}{t} - m^{-1}| > \delta$ states about the random variables T_i .

- HW₂** **5.7** Let $X_1, X_2, \dots$ be random variables over the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ having the same distribution function $F(x)$. (Nothing else is assumed.) Let $M_n := \max_{1 \leq i \leq n} |X_i|$. (a) Suppose that for some $\alpha > 0$, $\int |x|^\alpha dF(x) < \infty$ (i.e. $\mathbb{E}(|X_i|^\alpha) < \infty$). Prove that for every $\varepsilon > 0$ and $\delta > 0$

$$\lim_{n \rightarrow \infty} \mathbb{P}(n^{-(1/\alpha+\varepsilon)} |M_n| > \delta) = 0.$$

- (b) Suppose that for some $s > 0$, $\int e^{s|x|} dF(x) < \infty$ (i.e. $\mathbb{E}(e^{s|X_i|}) < \infty$). Prove that for any sequence b_n such that $\lim_{n \rightarrow \infty} b_n = \infty$ and for every $\delta > 0$

$$\lim_{n \rightarrow \infty} \mathbb{P}((b_n \log n)^{-1} |M_n| > \delta) = 0.$$

Hint: Use the following Markov-like inequality.

$$\mathbb{P}\left(\max_{1 \leq i \leq n} |X_i| > \lambda\right) = \mathbb{P}\left(\cup_{i=1}^n \{|X_i| > \lambda\}\right) \leq \sum_{i=1}^n \mathbb{P}(|X_i| > \lambda) = n\mathbb{P}(|X_1| > \lambda).$$

5.8 We toss a coin 60 times and denote the number of heads by X . Give an upper bound for the probability

$$\mathbb{P}(|X - 30| \geq 20)$$

by using Chebysev's inequality. A better estimate can be given by using the turbo-Markov inequality:

- (a) Let $Y_\beta = e^{\beta X}$, where $0 < \beta$. Show that $\mathbb{E}(Y_\beta) = 2^{-60}(1 + e^\beta)^{60}$.
- (b) Give an upper estimate for $\mathbb{P}(X \geq 50)$ by using Markov-inequality for the non-negative random variable Y_β for all $\beta > 0$.
- (c) Find the optimal β , that is, find the minimum of the estimate in (b). (This can be done by minimizing the convex function $f(\beta) = \log(1 + e^\beta) - \frac{5}{6}\beta$.)
- (d) Combining the previous points, show $\mathbb{P}(|X - 30| \geq 20) \leq 2 \cdot 3^{60} \cdot 5^{-50} < 10^{-6}$.

- HW 5.9** (a) Let X be a random variable. We call the function $R(t) = \mathbb{E}(e^{tX})$ the moment generating function of X . Show that for every $x \in \mathbb{R}$, $\mathbb{P}(X > x) \leq \inf_{t>0} R(t)e^{-tx}$.
- (b) Let X be a random variable with distribution $POI(\lambda)$. Using the exercise **5.9a**, estimate $\mathbb{P}(X > x)$.