Probability Theory 2

8th Exercise Sheet: Types of convergences 01.04.2025.

8.1 Let $X_1, X_2, \dots, X_n, \dots, Y_1, Y_2, \dots, Y_n, \dots, X$ and Y be random variables on the same prob. space $(\Omega, \mathcal{A}, \mathbb{P})$ and suppose that $X_n \xrightarrow{\mathbb{P}} X$ and $Y_n \xrightarrow{\mathbb{P}} Y$. Show
(a) $aX_n \xrightarrow{\mathbb{P}} aX$ $(a \in \mathbb{R})$,
(b) $X_n \pm Y_n \xrightarrow{\mathbb{P}} X \pm Y$,
(c) $X_n Y_n \xrightarrow{\mathbb{P}} XY$,
(d) If $\mathbb{P}(Y_n \neq 0) = \mathbb{P}(Y \neq 0) = 1$ then $X_n / Y_n \xrightarrow{\mathbb{P}} X/Y$.
(e) If $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f(X_n) \xrightarrow{\mathbb{P}} f(X)$.

8.2 Let $X_1, X_2, \dots$ and Y be random variables on the same prob. space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $F_1(x)$, $F_2(x)$, $\dots$ and $G(x)$ be their distribution functions respectively. Show that if $X_n \xrightarrow{\mathbb{P}} Y$ then $\lim_{n\to\infty} F_n(x) = G(x)$ at every continuity point x of G.

HW **8.3** Show that the convergence in probability is metrisable with respect to

$$\rho(X, Y) := \inf\{\varepsilon : \mathbb{P}(|X - Y| > \varepsilon) < \varepsilon\}.$$

That is, show that ρ is a metric and $X_n \xrightarrow{\mathbb{P}} Y$ if and only if $\rho(X_n, Y) \to 0$. (Remark: This metric is complete.)

8.4 Let H be a subset of L^1 random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that the set H is uniformly integrable if

$$\lim_{M\to\infty} \sup_{X \in H} \mathbb{E}\Big(|X| \mathbb{1}_{\{|X|>M\}}\Big) = 0.$$

(a) Show that every finite subset of $L^1(\Omega, \mathbb{P})$ is uniformly integrable.
(b) If $Y \in L^1$ then the set $\{X \in L^1(\Omega) : |X| < |Y|\}$ is uniformly integrable.
(c) If H is uniformly integrable then $\{X \in L^1(\Omega) : \exists Y \in H \ |X| \leq |Y|\}$ is unif. integrable.

HW* **8.5** Show that a set H of random variables is uniformly integrable if and only if the followings hold:
(1) $\sup_{X \in H} \mathbb{E}\,|X| < \infty$
(2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathbb{E}(|X|\, \mathbb{1}_A) \leq \varepsilon$ for every $\mathbb{P}(A) \leq \delta$ and every $X \in H$.

8.6 Let $f\colon [0,1] \mapsto \mathbb{R}$ be bounded, three times continuously differentiable function. Find the value of the next limit.

$$\lim_{n\to\infty} n \int_0^1 \cdots \int_0^1 \left(f\left(\frac{x_1 + \cdots + x_n}{n}\right) - f\left(\frac{1}{2}\right)\right) dx_1 \cdots dx_n$$

Hint: Approximate with Taylor polynomials.

8.7 Show that if $X_n \xrightarrow{L^1} X$ then $\mathbb{E}(X_n) \to \mathbb{E}(X)$. Furthermore, if $\mathbb{E}(X_n) \to \mathbb{E}(X)$ and $\mathbb{P}(X_n \leq X) = 1$ for every n then $X_n \xrightarrow{L^1} X$.

8.8 Find the value of the next limit.

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1\, dx_2 \, \cdots \, dx_n$$

HW **8.9** Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that

(a) $$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) dx_1\, dx_2 \,\cdots\, dx_n = f\left(\frac{1}{2}\right),$$

(b) $$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \cdots x_n)^{1/n}\right) dx_1\, dx_2 \,\cdots\, dx_n = f\left(\frac{1}{e}\right).$$

8.10 Let $X_1, X_2, \ldots, X_n, \ldots$ be random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $Y_n := \sup_{m\geq n} |X_m|$. Show that the following statements are equivalent:

(i) $X_n \xrightarrow{\text{a.s.}} 0$, as $n \to \infty$.

(ii) $Y_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$.

8.11 *Dominated convergence theorem* Let X_n be a sequence of random variables such that $X_n \xrightarrow{\text{a.s.}} X$ and there exists a random variable Y such that $\mathbb{E}(|Y|) < \infty$ and $\mathbb{P}(|X_n| \leq |Y|) = 1$ for every n. Then $\mathbb{E}(X_n) \to \mathbb{E}(X)$ as $n \to \infty$.

HW **8.12** *Scheffé's Theorem* Let X_n be a sequence of non-negative random variables and let X be a random variable such that $X_n \xrightarrow{\text{a.s.}} X$ and $\mathbb{E}(X_n) \to \mathbb{E}(X) < \infty$ as $n \to \infty$. Prove that $X_n \xrightarrow{L^1} X$.

(Hint: Use that $|Y| = Y + 2\max\{0, -Y\}$ and the dominated convergence theorem)