Probability Theory 2

9th Exercise Sheet: Borel-Cantelli lemmas & Kolmogorov's SLLN — 08.04.2025.

9.1 Let $X_1, X_2, \dots$ be independent random variables such that
$$\mathbb{P}\Big(X_n = n^2 - 1\Big) = n^{-2}, \quad \mathbb{P}\Big(X_n = -1\Big) = 1 - n^{-2}.$$
Prove that for every $n \in \mathbb{N}$, $\mathbb{E}\big(X_n\big) = 0$ but
$$\lim_{n\to\infty} \frac{X_1 + X_2 + \dots + X_n}{n} = -1 \quad \text{almost surely.}$$

9.2 Let X_n be i.i.d random variables with $X_n \sim GEO(p)$. That is $\mathbb{P}(X_n = k) = (1-p)p^k$ for $k \geq 0$. Show that $\limsup_{n\to\infty} \frac{X_n}{\log n} = |\log p|^{-1}$ almost surely.

9.3 We make infinitely many independent experiments. The probability that the nth experiment is successful is $n^{-\alpha}$, where $0 < \alpha < 1$. Let $k \geq 1$. It makes us happy if it happens infinitely often that we have k consecutive successful experiments. What is the probability that we are happy?

9.4 Let $X_1, X_2, \dots$ be independent random variables such that $\mathbb{P}(X_n = 1) = p_n$ and $\mathbb{P}(X_n = 0) = 1 - p_n$. Which properties does $p_n, n = 1, 2, \dots$ have if
(a) $X_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$
(b) $X_n \xrightarrow{\textbf{a.s.}} 0$ as $n \to \infty$.

9.5 Let $X_i \sim EXP(\lambda_i)$ be i.i.d. random variables. Give necessary and sufficient conditions for the following: (a) $X_n \xrightarrow{\mathbb{P}} 0$, (b) $X_n \xrightarrow{\textbf{a.s.}} 0$, (c) $X_n \xrightarrow{L^1} 0$.

HW **9.6** Let $X_1, X_2, \dots$ be independent random variables. Prove that $\sup\limits_n X_n < \infty$ with probability 1 if and only if $\sum_{n=1}^{\infty} \mathbb{P}(X_n > A) < \infty$ for some positive real number A.

9.7 Let $X_1, X_2, \dots, X_n, \dots$ be i.i.d non-negative random variables, which are not degenerated. (That is, $1 = \mathbb{P}(X_i \geq 0) \geq \mathbb{P}(X_i > 0) > 0$.) Show that the series $\sum\limits_{n=1}^{\infty} \mathbb{P}\Big(\sum\limits_{i=1}^{n} X_i < x\Big) < \infty$ is convergent.

9.8 Let $X_1, X_2, \dots, X_n, \dots$ be i.i.d random variables. Show that the following statements are equivalent:
(i) $\mathbb{E}|X_i| < \infty$.
(ii) $\mathbb{P}(|X_n| > n \;\text{ for infinitely many } n\;) = 0$.

9.9 Prove that for any sequence of random variables $X_1, X_2, \dots$ there exists a sequence of real numbers $c_1, c_2, \dots$ such that
$$\frac{X_n}{c_n} \xrightarrow{\textbf{a.s.}} 0.$$

9.10 Let $f : [0,1] \to \mathbb{R}$ be a continuous integrable (i.e. $\int_0^1 |f(x)|\, dx < \infty$) function. Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution UNI$(0,1)$. Show that
$$\mathbb{P}\left(\lim_{n\to\infty} \frac{f(X_1) + \dots + f(X_n)}{n} = \int_0^1 f(x)\, dx\right) = 1.$$

HW 9.11 *Simplest form of McMillan's Theorem.* Let $\mathbf{p} = (p_1, p_2, \dots, p_r)$ be such that p_i, $i = 1, 2, \dots, r$ are positive reals and $p_1 + p_2 + \cdots + p_r = 1$. That is, $\mathbf{p}$ is a probability distribution on $\{1, 2, \dots, r\}$. We call the quantity $H(\mathbf{p}) := -\sum_{j=1}^{r} p_j \log p_j$ the *entropy* of $\mathbf{p}$. Let $X_1, X_2, \dots$ be i.i.d random variables such that $\mathbb{P}(X_n = j) = p_j$ for $j \in \{1, \dots, r\}$. Let $R_n := \prod_{k=1}^{n} p_{X_k}$. The random variable R_n is the *a priori probability of the sequence* $X_1, X_2, \dots, X_n$. Show that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n} \log R_n = -H(\mathbf{p}) \right) = 1.$$

9.12 (Longest sequence of heads I.) Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution $\mathbb{P}(X_k = 1) = p$, $\mathbb{P}(X_k = 0) = q$, where $p + q = 1$. Let us fix a parameter $\lambda > 1$ and denote $A_k^{(\lambda)}$ for $k = 0, 1, 2, \dots$ the following event

$$A_k^{(\lambda)} := \left\{ \exists r \in \left[[\lambda^k], [\lambda^{k+1}] - k \right] \cap \mathbb{N} : X_r = X_{r+1} = \cdots = X_{r+k-1} = 1 \right\}.$$

In particular, the event $A_k^{(\lambda)}$ means that between $[\lambda^k]$ and $[\lambda^{k+1}] - 1$ there exists somewhere a sequence containing only 1 and with length k Show that
(a) If $\lambda < p^{-1}$ then $A_k^{(\lambda)}$ happen for at most finitely many k with probability 1.
(b) If $\lambda \geq p^{-1}$ then with probability 1, the events $A_k^{(\lambda)}$ happen for infinitely many k.
Hint: Use the Borel-Cantelli lemmas.

HW 9.13 (Longest sequence of heads II.) Let

$$R_n := \sup\{k \geq 0 : X_n = X_{n+1} = \cdots = X_{n+k-1} = 1\}.$$

That is, R_n denotes the length of sequnece of 1s beginning at n. (If $X_n = 0$ then $R_n = 0$.) Show that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{R_n}{\log n} = |\log p|^{-1} \right) = 1.$$

Hint: For every fixed parameter value of $\alpha > 0$, let

$$B_n^{(\alpha)} := \{R_n > \alpha \log n / |\log p|\}.$$

Show that if $\alpha > 1$ then with probability 1 only finitely many events $B_n^{(\alpha)}$. If $\alpha \leq 1$ then show that $B_n^{(\alpha)}$ happen for infinitely many n with probability 1.

HW* 9.14 *(Reverse of the Law of Large Numbers.)*

(a) Let Z be a non-negative random variable and let

$$Y := [Z] = \sum_{n=1}^{\infty} \mathbb{1}_{\{Z \geq n\}}.$$

Show that

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(Z \geq n\Big) \leq \mathbb{E}\Big(Z\Big) \leq 1 + \sum_{n=1}^{\infty} \mathbb{P}\Big(Z \geq n\Big).$$

(b) Let $X_1, X_2, \dots$ be i.i.d. random variables such that $\mathbb{E}\big(|X_n|\big) = \infty$. Prove that for every $M < \infty$

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(|X_n| \geq Mn \Big) = \infty \text{ and hence, } \mathbb{P}\left(\limsup_{n\to\infty} \frac{|X_n|}{n} = \infty \right) = 1.$$

(c) Let $S_n := X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \dots$ are random variables from (b). Show that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{|S_n|}{n} = \infty \right) = 1.$$