Probability Theory 2

10th Exercise Sheet: Kolmogorov's 0-1 Law 15.04.2025.

10.1 Show an example for a sequence of independent non-negative random variables $X_1, X_2, \dots$ such that the sum of the variances is divergent (that is $\sum_{i=1}^{\infty} \mathbb{D}^2(X_i) = \infty$) but the sequence $S_n = X_1 + \cdots + X_n$ converges almost surely to a finite random variable.

10.2 Let $X_1, X_2, \dots$ be uncorrelated random variables on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and expected value 0 respectively. Show that if the sum of the variances is finite then the sequence $S_n = X_1 + \cdots + X_n$ is convergent in L^2.

10.3 Let $X_1, X_2, \dots$ be independent random variables such that $X_n \sim EXP(\lambda_n)$ and let $S_n := \sum_{i=1}^{n} X_i$. Prove that if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$ then $S_n \xrightarrow{\text{a.s.}} \infty$, but if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$ then $S_n \xrightarrow{\text{a.s.}} S$, where S is a random variable with $\mathbb{P}(S < \infty) = 1$.

HW 10.4 Let θ_n be a sequence and let $X_1, X_2, \dots$ be a sequence of independent random variables such that $X_n \sim UNI[0, \theta_n]$. Show that the series $S_n = X_1 + \cdots + X_n$ converges almost surely to a finite random variable if and only if $\sum_{n=1}^{\infty} \theta_n < \infty$.

10.5 Let $X_1, X_2, \dots$ be independent random variables and let b_n be a sequence such that $b_n \nearrow \infty$.
(a) Prove that $\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} X_i \text{ exists}\right)$ is either 0 or 1.
(b) Suppose that $\frac{1}{b_n} \sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} Z$ for some random variable Z. Show that Z is constant almost surely.

10.6 Let $X_0, X_1, X_2, \dots$ be a sequence of independent random variables. Show that the radius of convergence of the random Taylor series $\sum_{k=0}^{\infty} X_k z^k$ is r with probability 1 for some non-random $r \in \mathbb{R}_+ \cup \{0, \infty\}$.

HW* 10.7 Suppose that $X_0, X_1, \dots$ are i.i.d. Show that the radius of convergence of the random Taylor series $\sum_{k=0}^{\infty} X_k z^k$ is 1 if $\mathbb{E}(\log^+ |X_i|) < \infty$ and 0 if $\mathbb{E}(\log^+ |X_i|) = \infty$ with probability 1, where $\log^+(x) = \max\{0, \log(x)\}$.

10.8 *(Not-iterated logarithm)* Let X be a random variable with standard normal distribution $N(0,1)$. We have shown earlier that for any $x > 0$

$$\left(\frac{1}{x} - \frac{1}{x^3}\right) \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \le \mathbb{P}\Big(X > x\Big) \le \frac{1}{x} \frac{\exp(-x^2/2)}{\sqrt{2\pi}}.$$

(a) Let $X_1, X_2, \dots$ be i.i.d with $X_i \sim N(0,1)$. Show that

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1\Big) = 1.$$

(b) Let $S_n := X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \dots$ are from (a). Show that for every $C > \sqrt{2}$

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n \log n}} < C\Big) = 1.$$

(*Note:* The Iterated logarithms Thm claims that $\mathbb{P}\Big(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = 1\Big) = 1$.)

HW$_2$ 10.9 Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution $POI(\mu)$.
(a) Let $Y_i = \mu - X_i$. Find the rate function $I(x)$ of Y_1!
(b) Show that for every $x \ge 0$, $I(x) \ge x^2/(2\mu)$!
(c) Let $S_n = X_1 + \cdots + X_n$. Show that

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{S_n - n\mu}{\sqrt{2\mu n \log n}} \ge -1\right) = 1$$