Probability Theory 2

11th Exercise Sheet: Characteristic functions I 29.04.2025.

11.1 (a) Show that $\varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$.

(b) Show if X, Y are independent the characteristic function ϕ_{X+Y} of $X+Y$ is the product of the characteristic functions ϕ_X, ϕ_Y.

(c) Show an example that $\phi_{X+Y} = \phi_X\phi_Y$ but X and Y are not independent.

11.2 Find the characteristic functions of the following distributions: $BIN(n,p)$, $GEO(p)$, $POI(\lambda)$, $EXP(\lambda)$, $N(\mu,\sigma^2)$, $CAU(a,b)$ and $UNI(a,b)$.

11.3 (a) Let ϕ be a characteristic function. Show that $\overline{\phi}, \phi^2, |\phi|^2$ are characteristic functions too.

(b) Let $\phi_1, \dots, \phi_n$ be characteristic functions. Show that $\sum_{i=1}^n q_i\phi_i$ is a characteristic function for any $q_1, \dots, q_n \geq 0$ with $\sum_{i=1}^n q_i = 1$. In fact, show that $\mathrm{Re}(\phi)$ is a characteristic function too.

(c) Show an example that $|\phi|$ is not a characteristic function.

11.4 Are the following functions characteristic function for some random variable?

$$\text{(a) } \frac{1}{1+t^2}, \quad \text{(b) } e^{-t^4}, \quad \text{(c) } \sin t, \quad \text{(d) } \cos t, \quad \text{(e) } \frac{1+\cos t}{2}.$$

11.5 Let $f(x) = 1 - |x|$ for $|x| \leq 1$, and $f(x) = 0$ for $|x| > 1$. Find the characteristic function of random variables with density f.

HW$_2$ 11.6 Find the characteristic functions of the random variables with densities:

$$\text{(a) } \frac{a}{2}e^{-a|x|}, x \in \mathbb{R}; \quad \text{(b) } \frac{|x|}{2}e^{-|x|}, x \in \mathbb{R}.$$

HW 11.7 Let $\varphi(t)$ be a characteristic function.
(a) Show that $\psi(t) := 1/(2 - \varphi(t))$ is a characteristic function too.
(b) Show that $\int_0^\infty \varphi(tu)e^{-u}du$ is a characteristic function as well.

HW 11.8 Show that a random variable X has symmetric distribution if and only if the characteristic function of X is real. (We call the distribution function F symmetric, if for every $x \in \mathbb{R}$, $F(-x) = 1 - F(x)$.)

11.9 Show an example of a random variable X such that φ_X is countinuously differentiable but $\mathbb{E}(|X|) = \infty$.

11.10 Let X be a random variable and let ϕ_X the characteristic function of X. Suppose that there exists a $t_0 \in \mathbb{R}$ such that $t_0 \neq 0$ and $|\phi_X(t_0)| = 1$. Show that X has lattice distribution. That is, there exists an arithmetic sequence $p_n = an + b$ with some $a \neq 0, b \in \mathbb{R}$ such that $\mathbb{P}(X \in \{p_n\}_{n\in\mathbb{Z}}) = 1$.

11.11 Show that the difference of two i.i.d random variable cannot have distribution $UNI(-1,1)$.

HW*11.12 Prove the identity

$$\frac{\sin t}{t} = \prod_{k=1}^\infty \cos\left(\frac{t}{2^k}\right)$$

by using characteristic functions.