Probability Theory 2

12th Exercise Sheet: Characteristic functions II 06.05.2025.

12.1 (a) Let U be a random variable with distribution $UNI[0,1]$, and let Y and Z be independent and independent of U with distribution $EXP(1)$. Show that $X = U(Y+Z)$ has distribution $EXP(1)$ as well.
(b) For which distributions F does it hold that if Y and Z are independent random variables with distribution F, and U is independent of Y and Z with distribution $UNI(0,1)$ then $X = U(Y+Z)$ has distribution F?

HW 12.2 Let X and Y be independent random variables with distribution $N(0,1)$, moreover, let U, V be independent and independent of X and Y with distribution $UNI(0,1)$. Show that the distribution of $\frac{UX+VY}{\sqrt{U^2+V^2}}$ is standard normal.

12.3 Let Y be a random variable with distribution $N(0,1)$. Show that the characteristic function $\varphi(t)$ of $X := |Y|$ is not real for every $t \in \mathbb{R} \setminus \{0\}$.

HW 12.4 Let $X_1, X_2, \ldots, X_n$ independent normal random variables with variance 1 and $\mathbb{E}X_i = \mu_i$, and let $Y = X_1^2 + X_2^2 + \cdots + X_n^2$. Show that the characteristic function of Y is

$$\varphi_Y(t) = \frac{1}{(1-2it)^{n/2}} \exp\left(\frac{it\theta}{1-2it}\right),$$

where $\theta := \sum_i^n \mu_i^2$. (We call the distribution of Y *non-centered* χ^2 *distribution with parameters* (n,θ).)

(*Hint:* Use induction.)

12.5 Let X be the log-normal distribution, that is, $X = e^Y$, where $Y \sim N(0,1)$.

(a) Show that the moments of X are $e^{k^2/2}$, and find its density function $f_X(x)$.
(b) Show that the function $g_{n,a}(x) := (1 + a\sin(2\pi n\log(x)))f_X(x)$ is a density function for every $a \in (-1,1)$ and $n \in \mathbb{N}$.
(c) Show that the moments of $g_{n,a}$ are $e^{k^2/2}$ for every $a \in (-1,1)$ and $n \in \mathbb{N}$.

12.6 (a) Show by using a probabilistic approach that $\int_{-\infty}^{\infty} \frac{\sin^2(t)}{t^2} dt = \pi$

(b) Show that $\frac{1}{\pi}\int_0^\infty \frac{\sin(at)}{t} dt = \begin{cases} \frac{1}{2} & \text{if } a > 0, \\ -\frac{1}{2} & \text{if } a < 0, \\ 0 & \text{if } a = 0. \end{cases}$

(c) *(Inversion formula II)* Let φ be the characteristic function of the random variable X with distribution function F. Then show that

$$\mathbb{P}(a < X < b) + \frac{\mathbb{P}(X=a) + \mathbb{P}(X=b)}{2} = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) dt.$$

HW 12.7 Show that if $X_1, X_2, \ldots, X_n$ are independent and uniformly distributed on $(-1,1)$ then for $n \geq 2$ the random variable $X_1 + \cdots + X_n$ has density

$$f(x) = \frac{1}{\pi} \int_0^\infty (\sin(t)/t)^n \cos(tx) dt.$$

12.8 (a) Let X be a random variable with characteristic function φ. Show that

$$\mathbb{P}(X = a) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\varphi(t)dt.$$

(b) Find the value of $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2 dt$.

HW 12.9 (a) Let X be a random variable with characteristic function φ. Suppose that $\mathbb{P}(X \in h\mathbb{Z}) = 1$, with some $h > 0$. Show that

$$\mathbb{P}(X = a) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-ita}\varphi(t)dt \text{ for } a \in h\mathbb{Z}.$$

(b) Suppose now that $\mathbb{P}(Y \in b + h\mathbb{Z}) = 1$ with $b \in [0, h)$. Using part (a), show that the previous formula holds for Y with $a \in b + h\mathbb{Z}$.

(Hint: Use exercise **12.8**(a).)

12.10 Let X and Y be i.i.d random variables with expected value 0 and variance 1. Denote the common characteristic function by φ. Suppose that $X + Y$ and $X - Y$ are independent. Show that this is possible only if X and Y are standard normal random variables.

Hint: Show that in this case $\varphi(2t) = \varphi(t)^3\varphi(-t)$, and show that this is possible only if $\varphi(t) = e^{-t^2/2}$. Step 1: $\varphi(t) = \varphi(-t)$ and step 2 $\varphi(t/2^n)^{4^n} \to e^{-t^2/2}$.

HW*12.11 *Invertability of the Laplace-transformation.* Let $f\colon [0,\infty) \mapsto \mathbb{R}$ be a continuous and bounded map. We define the Laplace-transform $\hat{f}$ of f as

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda y} f(y)dy.$$

(a) Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution $EXP(\lambda)$, i.e. $\mathbb{P}(X_k > x_k) = e^{-\lambda x}$; $x > 0$, $\mathbb{E}(X_k) = \lambda^{-1}$, and $\mathbb{D}^2(X_k) = \lambda^{-2}$. Show that

$$\mathbb{E}(f(S_n)) = (-1)^{n-1}\frac{\lambda^n \hat{f}^{(n-1)}(\lambda)}{(n-1)!},$$

where $S_n = X_1 + \cdots + X_n$ and $\hat{f}^{(n-1)}$ denotes the $n-1$th derivative of $\hat{f}$.

(b) Prove that the Laplace transform satisfies the following inversion formula (for all continuous and bounded map f):

$$f(y) = \lim_{n\to\infty} (-1)^{n-1}\frac{(n/y)^n \hat{f}^{(n-1)}(n/y)}{(n-1)!}.$$