

Probability Theory 2

13th Exercise Sheet: Weak convergence

13.05.2025.

13.1 Let $S = [0, 1]$ and let μ_n be the discrete measure such that every point k/n has weight $1/(n+1)$ for $k = 0, 1, \dots, n$. Show that $\mu_n \Rightarrow \mu$, where μ is the Lebesgue measure on S .

HW 13.2 Let X_n be a random variable with distribution $GEO(\frac{\lambda}{n})$ (that is, $\mathbb{P}(X_n = k) = (\lambda/n)(1 - \lambda/n)^k$, $k \geq 0$). Using the definition of the weak convergence, show that $\frac{X_n}{n}$ has limiting distribution and find it.

13.3 (*Cramer-Slutsky theorem*) Let X_n, Y_n, Z_n , $n \in \mathbb{N}$ and X be random variables over the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{\mathbb{P}} y$, $Z_n \xrightarrow{\mathbb{P}} z$ for some $y, z \in \mathbb{R}$ then $X_n Y_n + Z_n \xrightarrow{d} X \cdot y + z$.

HW 13.4 Let $F_n, F : \mathbb{R} \rightarrow [0, 1]$ be cumulative distribution functions. Assume that $F_n \Rightarrow F$ and F is continuous. Show that for any convergent sequence $x_n \rightarrow x$ of real numbers we have $F_n(x_n) \rightarrow F(x)$.

Limit theorems for maximum of i.i.d random variables.

Let $X_1, X_2, \dots$ be i.i.d random variables with distribution function $F(x) := \mathbb{P}(X_i < x)$. Let $M_n := \max\{X_1, X_2, \dots, X_n\}$. The asymptotic behavior of the random variable M_n as $n \rightarrow \infty$ depends on then asymptotic of the upper tail of $F(x)$. Show the following limit theorems (next 3 exercises):

13.5 Suppose that $F(x) < 1$ for every $x < \infty$ and $\lim_{x \rightarrow \infty} x^\alpha (1 - F(x)) = b$ for some $\alpha, b \in (0, \infty)$ Show that the distribution of $n^{-1/\alpha} M_n$ converges weakly to:

$$\mathbb{P}(n^{-1/\alpha} M_n < x) \rightarrow \mathbb{1}_{\{x > 0\}} \exp(-bx^{-\alpha}).$$

HW₂ 13.6 Suppose that $F(x_0) = 1$ and $F(x) < 1$ for every $x < x_0$, moreover, $\lim_{x \rightarrow x_0} (x_0 - x)^{-\alpha} (1 - F(x)) = b$ for some $\alpha, b \in (0, \infty)$ Then the distribution of $n^{1/\alpha} (x_0 - M_n)$ converges weakly to:

$$\mathbb{P}(n^{1/\alpha} (x_0 - M_n) < x) \rightarrow \mathbb{1}_{\{x > 0\}} (1 - \exp(-bx^\alpha)).$$

13.7 Suppose that $F(x) < 1$ for every $x < \infty$ and $\lim_{x \rightarrow \infty} e^{\lambda x} (1 - F(x)) = b$ for some $\lambda, b \in (0, \infty)$ Then the distribution of $M_n - \lambda^{-1} \log n$ converges weakly to:

$$\mathbb{P}(M_n - \lambda^{-1} \log n < x) \rightarrow \exp(-be^{-\lambda x}).$$

13.8 Let $X_1, X_2, \dots$ be i.i.d. random variables with standard normal distribution. From previous exercise **6.8** we know that

$$\mathbb{P}(X_i > x) \sim \frac{1}{x} e^{-x^2/2} \text{ as } x \rightarrow \infty.$$

(i) Show that for any real number θ

$$\frac{\mathbb{P}(X_i > x + \theta/x)}{\mathbb{P}(X_i > x)} \rightarrow e^{-\theta} \text{ as } x \rightarrow \infty.$$

(ii) Define b_n such that $\mathbb{P}(X_i > b_n) = 1/n$. Show that

$$\mathbb{P}(b_n(M_n - b_n) \leq x) \rightarrow e^{-e^{-x}} \text{ as } n \rightarrow \infty,$$

where $M_n = \max_{1 \leq m \leq n} X_m$.

(iii) Show that $b_n \sim (2 \log n)^{1/2}$ and conclude that $M_n / (2 \log n)^{1/2} \xrightarrow{\mathbb{P}} 1$.