

Probability Theory 2

14th Exercise Sheet: Method of characteristic functions 20.05.2025.

- 14.1** Let $X_1, X_2, \dots$ be i.i.d random variables with distribution $UNI(-1, 1)$. Find the limiting distribution of the sequence

$$Y_n := \frac{X_1 + \dots + X_n}{\sqrt{X_1^2 + \dots + X_n^2}}.$$

- 14.2** We toss a fair coin consecutively until the number of heads reaches n . Denote ν_n the number of required tosses. Find the limit distribution of $(\nu_n - 2n)/\sqrt{2n}$ as $n \rightarrow \infty$.

- 14.3** Let $X_n \sim POI(n)$ be independent random variables. Show that

$$\frac{X_n - n}{\sqrt{X_n}} \xrightarrow{d} N(0, 1) \text{ as } n \rightarrow \infty.$$

- 14.4** Let $X_1, X_2, \dots$ be non-negative i.i.d random variables with $\mathbb{E}(X_i) = 1$ and $\mathbb{D}^2(X_i) = \sigma^2$. Show that

$$2(\sqrt{S_n} - \sqrt{n}) \xrightarrow{d} N(0, \sigma^2).$$

- 14.5** Let $X_1, X_2, \dots$ be i.i.d random variables with $\mathbb{E}(X_i) = 0$ and $\mathbb{D}^2(X_i) = 1$. Use CLT and Kolmogorov's 0-1 law to show $\limsup_{n \rightarrow \infty} S_n/\sqrt{n} = \infty$ almost surely.

- 14.6** Prove the weak law of large numbers by using the method of characteristic functions.

- 14.7** Let X_n be random variables with normal distribution. Suppose that there exists a random variable X such that $X_n \xrightarrow{d} X$. Show that X has normal distribution, or it is degenerated (i.e. it is a constant with probability 1).

- 14.8** Show with the method of characteristic functions that $BIN(n, p_n) \Rightarrow POI(\lambda)$ as $n \rightarrow \infty$ if $\lim_{n \rightarrow \infty} np_n = \lambda$.

- 14.9** Let $X_p \sim GEO(p)$. Show that the limiting distribution of $p \cdot X_p$ is $EXP(1)$ as $p \rightarrow 0+$.

- 14.10** Let X be a random variable with distribution $GAM(1, s)$ (i.e. its density is $\frac{e^{-x}x^{s-1}}{\Gamma(s)}\mathbb{1}_{\{x>0\}}$), and let Y be a random variable with distribution $POI(X)$. Show that

$$\frac{Y - \mathbb{E}Y}{\mathbb{D}(Y)} \xrightarrow{d} N(0, 1) \text{ as } s \rightarrow \infty.$$

- 14.11** Let $X_1, X_2, \dots$ be a sequence of independent but uniformly bounded random variables (i.e. there exists $K \geq 1$ such that $\mathbb{P}(|X_i| \leq K) = 1$ for every $i \geq 1$) such that $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = \sigma_i^2$, where $B_n^2 = \sum_{k=1}^n \sigma_k^2 \rightarrow \infty$ as $n \rightarrow \infty$. Show that

$$\frac{X_1 + \dots + X_n}{B_n} \xrightarrow{d} N(0, 1).$$

- 14.12** Let $X_n \sim POI(n)$, and $Y_m \sim POI(m)$ be independent random variables for $n, m \in \mathbb{Z}_+$. Show that

$$\frac{X_n - n - (Y_m - m)}{\sqrt{X_n + Y_m}} \xrightarrow{d} N(0, 1) \text{ as } n, m \rightarrow \infty.$$

- 14.13** Show that

$$e^{-n} \left(1 + n + \frac{n^2}{2!} + \dots + \frac{n^n}{n!} \right) \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty.$$

14.14 The ‘coupon collector’s problem’:

In an urn there are n many balls, enumerated from 1 to n . We pull out the balls and after checking the number on it we return it into the urn. We continue this process until we pull out all of the balls at least once. Denote U_n the number of required pulls.

(a) Find $\mathbb{E}(U_n)$ and $\mathbb{D}^2(U_n)$. (We have already calculated it)

(b) Show that $U_n/(n \log n) \xrightarrow{\mathbb{P}} 1$ as $n \rightarrow \infty$. (We did this too.)

(c) Find the characteristic function of the distribution $F(x) = e^{-e^{-x}}$. (The moment generating function is easier and then substitute.)

(d) Show that $(U_n - n \log n)/n \xrightarrow{d} F$. (For (d) use (c), and the limit $\lim_{n \rightarrow \infty} \frac{(n-1)!n^z}{\prod_{r=0}^{n-1}(z+r)} = \Gamma(z)$. The limit holds uniformly on regions where the function Γ is regular.)

14.15 Let $\tau_1, \tau_2, \dots$ be i.i.d. non negative random variables with $\mathbb{E}(\tau_i) = m \in (0, \infty)$ and $\mathbb{D}^2(\tau_i) = \sigma^2 \in (0, \infty)$. Let $\nu_t := \max\{n : \sum_{i=1}^n \tau_i < t\}$.

We have already seen the WLLN:

$$\frac{\nu_t}{t} \xrightarrow{\mathbb{P}} m^{-1} \text{ as } t \rightarrow \infty.$$

Show the following limiting distribution:

$$\lim_{t \rightarrow \infty} \mathbb{P}\left(\frac{\nu_t - m^{-1}t}{\sigma \sqrt{m^{-3}t}} < x\right) = \Phi(x),$$

where Φ is the distribution function of $N(0, 1)$.

Hint: The CLT says that

$$\frac{\sum_{i=1}^n \tau_i - nm}{\sqrt{n}\sigma} \xrightarrow{d} N(0, 1).$$