Probability Theory 2

Sample exam — A long time ago in a galaxy far, far away

Ex.1 (a) *(5 points)* Define Euler's Gamma function $\Gamma(\alpha)$ for $\alpha > 0$!

(b) *(5 points)* State and prove the induction formula for Euler's Gamma function!

(c) *(5 points)* Define the gamma distribution with parameters $\alpha, \lambda > 0$!

(d) *(5 points)* State and prove the property of gamma distributions under convolution!

Ex.2 Let $X_1, X_2, \ldots$ be a sequence of random variables and let X be a random variable jointly defined over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) *(5 points)* When do we say that X_n converges to X in probability ($X_n \xrightarrow{\mathbb{P}} X$)?

(b) *(5 points)* When do we say that X_n converges to X almost surely ($X_n \xrightarrow{\textbf{a.s.}} X$)?

(c) *(5 points)* Does the convergence in probability imply almost sure convergence? (If yes prove it if no show a counterexample!)

(d) *(5 points)* Does the almost sure convergence imply convergence in probability? (If yes prove it if no show a counterexample!)

Ex.3 *(15 points)* State and prove both Borel Cantelli lemmas!

Ex.4 Let X be a random variable with distribution function $F\colon \mathbb{R} \to [0,1]$.

(a) *(5 points)* Define the characteristic function of X!

(b) *(10 points)* Show that the characteristic function is uniformly continuous and is positive definite!

Ex.5 *(15 points)* Let X, Y, Z be independent random variables such that X and Y have distribution $\mathrm{EXP}(1)$ and let Z be such that $\mathbb{P}(Z = -1) = \mathbb{P}(Z = 1) = \frac{1}{2}$. Show that the random variables $U := X - Y$ and $V := ZX$ have the same distribution!

Ex.6 *(15 points)* Let $Z_1, Z_2, \ldots$ be i.i.d. random variables with standard Cauchy distribution $\mathrm{CAU}(0,1)$. (The characteristic function of a standard Cauchy is $u \mapsto e^{-|u|}$.) Show that for any $\varepsilon > 0$

$$\frac{Z_1 + \cdots + Z_n}{n^{2+\varepsilon}} \xrightarrow{\textbf{a.s.}} 0 \qquad \text{as } n \to \infty!$$