Probability Theory 2

II. Midterm test solutions

MT.1 (a) First we show that $X_n \xrightarrow{\mathbb{P}} 0$. Let $\delta > 0$, then using Markov's inequality

$$\mathbb{P}\left(|X_n| > \delta\right) = \mathbb{P}\left(X_n > \delta\right) \leq \frac{\mathbb{E}\left(X_n\right)}{\delta} = \frac{1}{\delta \log(n)} \to 0 \quad \text{as } n \to \infty.$$

Now we show that $X_n \xrightarrow{L^2} 0$.

$$\mathbb{E}\left(X_n^2\right) = \mathbb{D}^2\left(X_n\right) + \mathbb{E}\left(X_n\right)^2 = \frac{2}{\log^2(n)} \to 0 \quad \text{as } n \to \infty.$$

(b) Let $c > 0$, then notice

$$\mathbb{P}\left(X_n \geq c\right) = 1 - \mathbb{P}\left(X_n < c\right) = 1 - \left(1 - e^{-\log(n)c}\right) = \frac{1}{n^c}.$$

Therefore

$$\sum_{k=2}^{\infty} \mathbb{P}\left(X_n \geq c\right) = \sum_{k=2}^{\infty} \frac{1}{n^c} \begin{cases} = \infty & \text{if } c \leq 1 \\ < \infty & \text{if } c > 1 \end{cases}$$

From the first Borel-Cantelli lemma it follows for any $c > 1$

$$\mathbb{P}\left(X_n \geq c \text{ for infinitely many } n\right) = 0.$$

Since X_n are independent from the second Borel-Cantelli lemma it follows for any $0 < c \leq 1$

$$\mathbb{P}\left(X_n \geq c \text{ for infinitely many } n\right) = 1.$$

The two imply

$$\mathbb{P}\left(\limsup_{n\to\infty} X_n = 1\right) = 1.$$

MT.2 First notice

$$\mathbb{P}\left(0 \leq Y \leq 1\right) = 1.$$

Moreover let $x \in [0,1]$. Then

$$\mathbb{P}\left(Y < x\right) = \mathbb{P}\left(\max\left\{X_1, X_2\right\} < x\right) = \mathbb{P}\left(X_1 < x, X_2 < x\right) = \mathbb{P}\left(X_1 < x\right)\mathbb{P}\left(X_2 < x\right) = x^2.$$

Therefore

$$\mathbb{P}\left(Y < x\right) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

It follows Y is absolute continuous with density

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then the characteristic function of Y is

$$\mathbb{E}\left(e^{itY}\right) = \int_0^1 e^{itx} 2x \, dx = \left[\frac{e^{itx} 2x}{it}\right]_{x=0}^{x=1} - \int_0^1 \frac{2e^{itx}}{it} \, dx = \frac{e^{it}\left(2 - 2it\right) - 2}{t^2}.$$