

Auxiliary Sheet

Generating functions

• Probability generating function

Let X be an \mathbb{N} -valued random variable. Its probability generating function is $P : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$P(z) = \sum_{n=0}^{\infty} z^n \mathbb{P}(X = n) = \mathbb{E}(z^X).$$

The generating function of the sequence (a_n) in general is $z \mapsto \sum_{n=0}^{\infty} a_n \cdot z^n$ (for $z \in \mathbb{C}$ inside the radius of convergence).

Some probability generating functions

$$\begin{array}{l|l} \text{Bin}(n, p) & ((1-p) + pz)^n \\ \text{Geo}(p) & p/(1 - (1-p)z) \\ \text{Poi}(\lambda) & \exp(\lambda(z-1)) \end{array}$$

• Properties of the PGF

- P is well-defined for $|z| \leq 1$ and analytic on $|z| < 1$.
- The derivatives of P on $[0, 1]$ are non-negative. Therefore, it is monotone increasing and convex on $[0, 1]$.
- The probability of the finiteness of X can be given by

$$\lim_{z \rightarrow 1} P(z) = \mathbb{P}(X < \infty).$$

- The factorial moments of X can be given by the derivatives of P . That is That is

$$\lim_{z \nearrow 1} P^{(n)}(z) = \mathbb{E}\left(\prod_{k=0}^{n-1} (X - k)\right), \quad n \geq 1.$$

For $n = 1, 2$ it follows

$$\mathbb{E}(X) = \lim_{z \nearrow 1} P'(z) \quad \text{and} \quad \mathbb{E}(X(X-1)) = \lim_{z \nearrow 1} P''(z).$$

• Independent variables and PGFs

- Let X and Y be two independent random variables with PGFs P and G . Then the PGF of $X + Y$ is

$$z \mapsto P(z)G(z), \quad |z| \leq 1.$$

- Let X_1, X_2, \dots be i.i.d. random variables with PGF P . Let $\nu \geq 1$ be independent of them with PGF G . Then the PGF of the random sum

$$\sum_{k=1}^{\nu} X_k$$

is

$$z \mapsto G(P(z)).$$

- Let X_1, X_2, \dots be independent with PGFs P_1, P_2, \dots . Let $\xi \geq 1$ be an almost surely finite \mathbb{N} -valued random variable independent from them. Then the PGF of the random element

$$X_{\xi}$$

is

$$z \mapsto \sum_{n=1}^{\infty} P_n(z) \mathbb{P}(\xi = n).$$