Auxiliary Sheet

Characteristic functions

- **Characteristic function**
  Let X be a real-valued random variable with distribution F. The characteristic function of X is
  $$\phi : \mathbb{R} \mapsto \mathbb{C}, \qquad \phi(u) := \mathbb{E}\Big(e^{iuX}\Big) = \int_{-\infty}^{\infty} e^{iux}\,\mathrm{d}F(x).$$
- **Some characteristic functions**

$\mathrm{Bin}(n,p)$	$u \mapsto \left(1-p+pe^{iu}\right)^n$	$\mathrm{Uni}(a,b)$	$u \mapsto \frac{e^{iub}-e^{iua}}{iu(b-a)}$
$\mathrm{Poi}(\lambda)$	$u \mapsto \exp\left(\lambda\left(e^{iu}-1\right)\right)$	$\mathcal{N}(\mu,\sigma^2)$	$u \mapsto \exp\left(iu\mu - \frac{u^2\sigma^2}{2}\right)$
$\mathrm{Geo}(p)$	$u \mapsto \frac{p}{1-(1-p)e^{iu}}$	$\mathrm{Cau}(a,b)$	$u \mapsto \exp\left(iua - b\,\lvert u\rvert\right)$
$\mathrm{Exp}(\lambda)$	$u \mapsto \frac{\lambda}{\lambda - iu}$		

- **Properties of the characteristic function**
  (i) $\phi(0) = 1$
  (ii) $\lvert\phi(u)\rvert \le 1$
  (iii) $u \mapsto \phi(u)$ is uniformly continuous
  (iv) $u \mapsto \phi(u)$ is of positive type. I.e. for any $n \in \mathbb{N}^+$ and $z_1, \dots z_n \in \mathbb{C}$, $u_1, \dots, u_n \in \mathbb{R}$
  $$\sum_{k=1}^{n}\sum_{l=1}^{n} z_k \overline{z_l}\,\phi\left(u_k - u_l\right) \ge 0.$$
- **Bochner's theorem**
  A function $\phi : \mathbb{R} \to \mathbb{C}$ is a characteristic function if and only if the following properties hold:
  (i) $\phi(0) = 1$
  (ii) $u \mapsto \phi(u)$ is continuous at $u = 0$
  (iii) $u \mapsto \phi(u)$ is of positive type
- **Moments**
  Suppose for some $n \in \mathbb{N}$ $\mathbb{E}(\lvert X^n\rvert) < +\infty$. Then:
  (i) $u \mapsto \phi(u)$ is C^n
  (ii) for any $0 \le k \le n$ $\phi^{(k)}(0) = i^k\,\mathbb{E}\Big(X^k\Big)$
  (iii) the Taylor expansion at $u = 0$ is
  $$\phi(u) = \sum_{k=0}^{n} \mathbb{E}(X^k)\frac{(iu)^k}{k!} + o(\lvert u\rvert^n)$$
- **Inversion formula**
  Let X be a random variable with cumulative distribution function F and characteristic function ϕ. If the real numbers a, b are points of continuity of F, then
  $$F(b) - F(a) = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-iua}-e^{-iub}}{iu}\phi(u)\,du,$$
  and
  $$F(b) - F(a) = \lim_{\sigma\to 0_+}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iua}-e^{-iub}}{iu}e^{-\frac{\sigma^2u^2}{2}}\phi(u)\,du.$$
- **Inversion formula (in case of absolute continuity)**
  Let X be a random variable with probability density function f and characteristic function ϕ. If $\int_{-\infty}^{\infty}\lvert\phi(u)\rvert\,du < \infty$ (in particular, if $f \in C^2$), then for any $x \in \mathbb{R}$
  $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi(u)\,du.$$
- **The characteristic function uniquely determines the distribution**
  Let X and Y be random variables with characteristic functions ϕ_X and ϕ_Y. Then the following are equivalent:
  (i) $X \overset{d}{=} Y$
  (ii) for all $u \in \mathbb{R}$ $\phi_X(u) = \phi_Y(u)$