Auxiliary Sheet

Method of characteristic functions and the Central Limit Theorem

- **Lévy's continuity theorem**
  Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables with characteristic functions $(\phi_n)_{n=1}^{\infty}$. If
  (i) for every $u \in \mathbb{R}$
  $$\lim_{n\to\infty} \phi_n(u) =: \phi(u) \quad \text{exists,}$$
  (ii) $u \mapsto \phi(u)$ is continuous at $u = 0$,

  then there exists a random variable X with characteristic function ϕ such that
  $$X_n \xrightarrow{d} X.$$

- **Method of characteristic functions**
  Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables with characteristic functions $(\phi_n)_{n=1}^{\infty}$ and let X be a random variable with characteristic function ϕ. The following are equivalent.
  (i) $X_n \xrightarrow{d} X$
  (ii) For any $u \in \mathbb{R}$
  $$\lim_{n\to\infty} \phi_n(u) = \phi(u).$$

- **The Central Limit Theorem**
  Let $X_1, X_2, \dots$ be independent and identically distributed random variables with finite second moment $\mathbb{E}(X_1^2) < \infty$. Denote $\mu := \mathbb{E}(X_1)$, $\sigma^2 = \mathbb{D}^2(X_1)$ and $S_n = X_1 + \cdots + X_n$. Then
  $$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$