1st Exercise Sheet
Convolutions I

1.1 Let X and Y be independent random variables with distribution

(a) $\mathrm{Bin}(n, p)$ and $\mathrm{Bin}(m, p)$ respectively, where $0 < p < 1$ and $n, m \in \mathbb{N}$;

(b) $\mathrm{Poi}(\lambda)$ and $\mathrm{Poi}(\mu)$ respectively, where $\lambda, \mu > 0$;

What is the distribution of $X + Y$?

HW 1.2 Give the distribution of the convolution of n independent identically distributed random variables with distribution $\mathrm{Geo}(p)$ (having probability mass function $k \mapsto p(1-p)^k,\ k \geq 0$)!

1.3 May B. Dunn is a student in mathematics on BUTE. She tries to pass the Probability Theory 2 course. First, she needs to get the signature in the practical part. If she fails in one semester, she tries again in the next one. The semesters are independent and in each of the semesters the probability that she gets the signature is 1/3. If she gets the signature she will try the oral exam on the theory. Again, if she fails she tries in the next semester, the semesters are independent and in each of the semesters, the probability that she passes the oral exam is 1/4. Find the distribution of the number of semesters required for May B. Dunn to pass.

1.4 Let X and Y be independent random variables with distribution $\mathrm{Exp}(\lambda)$ and $\mathrm{Exp}(\mu)$. Find the density of $Z := X + Y$.

HW$_2$ 1.5 Generalize the exercise **1.4**: Determine the density function of the random variable $X_1 + X_2 + \cdots + X_n$, where X_i are independent random variables with distribution $\mathrm{Exp}(\mu_i)$ for every $i = 1, \ldots, n$, and the parameters μ_i are pairwise different. (Hint: Show that for the polynomial $P_n(x) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - \mu_j}{\mu_i - \mu_j} = 1$ for every $x \in \mathbb{R}$. Check the values at $\mu_1, \ldots, \mu_n$ and the degree of P_n. Use this fact.)

HW* 1.6 A player at time $t = 0$ starts a game with infinitely many levels. He needs $\mathrm{Exp}(1)$ time to finish the first level. After, he gets better in the game and only needs $\mathrm{Exp}(2)$ time to finish the second level. Similarly, he needs $\mathrm{Exp}(k)$ time to finish the kth level independently from the previous levels. Find the distribution of the completed levels by time $t > 0$! (Hint: Use the result of **1.5**.)

1.7 Let X and Y be i.i.d random variables with common density function $f(x) = 2x\mathbb{1}_{[0,1]}(x)$. Find the density functions of $U := X + Y$ and $V := X - Y$.

1.8 Let X_1, X_2 and X_3 be i.i.d random variables with distribution $\mathrm{Uni}(0, 1)$. Find the density functions of the random variables $Y := X_1 + X_2$ and $Z := X_1 + X_2 + X_3$.

1.9 Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d random variables with distribution $\mathrm{Uni}(0, 1)$. Denote $f_n(x)$ the density function of $S_n := \sum_{k=1}^{n} X_k$. Prove that

$$f_n(x) = \frac{1}{(n-1)!} \sum_{k=0}^{\lfloor x \rfloor} (-1)^k \binom{n}{k} (x-k)^{n-1}.$$

Using a computer program (e.g. *Mathematica, Python*) plot the graph of the function

$$\widetilde{f}_n(x) := \sqrt{\frac{n}{12}} f_n\Big(\frac{n}{2} + \sqrt{\frac{n}{12}} x\Big)$$

for $n = 1, 2, \ldots, 10$. What do we see? Interpret the result!

1.10 Let X and Y be independent r.v. with dist. $\mathrm{Poi}(\lambda)$ and $\mathrm{Uni}(0, 1)$. Find the distribution of $Z := X + Y$.

HW 1.11 Let X and Y be independent r.v. with dist. $\mathrm{Geo}(p)$ and $\mathrm{Uni}(0, 1)$. Find the distribution of $Z := X + Y$.

1.12 Let X be a random variable with distribution $\mathrm{Uni}(0, 1)$, and let Y be an arbitrary random variable independent of X. Prove that the random variable $Z := \{X + Y\} := (X + Y) - \lfloor X + Y \rfloor$ has distribution $\mathrm{Uni}(0, 1)$, independently of the distribution of Y.