# 1st Exercise Class
## Convolutions I

**1.1** Let $X$ and $Y$ be independent random variables with distribution

(a) $\text{Bin}(n,p)$ and $\text{Bin}(m,p)$ respectively, where $0<p<1$ and $n,m\in\mathbb{N}$;

What is the distribution of $X+Y$?

**Solution**

(a) For $0\le k\le n+m$

$$\begin{aligned}\mathbb{P}(X+Y=k) &= \sum_{l=0}^{\infty}\mathbb{P}(X=k-l,Y=l)\\ &= \sum_{l=0}^{k}\mathbb{P}(X=k-l)\mathbb{P}(Y=l)\\ &= \sum_{l=0}^{k}\binom{n}{k-l}p^{k-l}(1-p)^{n-(k-l)}\binom{m}{l}p^{l}(1-p)^{m-l}\\ &= p^k(1-p)^{(n+m)-k}\sum_{l=0}^{k}\binom{n}{k-l}\binom{m}{l}\\ &= p^k(1-p)^{(n+m)-k}\binom{n+m}{k}.\end{aligned}$$

Therefore, $X+Y\sim\text{Bin}(n+m,p)$.

**1.4** Let $X$ and $Y$ be independent random variables with distribution $\text{Exp}(\lambda)$ and $\text{Exp}(\mu)$. Find the density of $Z:=X+Y$.

**Solution** The PDFs are $f(x)=\lambda e^{-\lambda x}\mathbb{1}[0,\infty)(x)$ and $g(x)=\mu e^{-\mu x}\mathbb{1}[0,\infty)(x)$. Let $h(x)$ denote the density of $Z$. Then for $x\le 0$ we have $h(x)=0$ and for $x>0$

$$h(x)=\int_{-\infty}^{\infty}f(y)g(x-y)\,\mathrm{d}y=\int_0^x f(y)g(x-y)\,\mathrm{d}y=\lambda\mu e^{-\mu x}\int_0^x e^{-(\lambda-\mu)y}\,\mathrm{d}y$$

Let us consider the following two cases.

(i) $\lambda=\mu$

$$h(x)=\lambda^2e^{-\lambda x}\int_0^x 1\,\mathrm{d}y=\lambda^2xe^{-\lambda x}$$

(ii) $\lambda\ne\mu$

$$h(x)=\lambda\mu e^{-\mu x}\int_0^x e^{-(\lambda-\mu)y}\,\mathrm{d}y=\lambda\mu e^{-\mu x}\left[\frac{e^{-(\lambda-\mu)y}}{\mu-\lambda}\right]_{y=0}^{y=x}=\frac{\lambda\mu}{\mu-\lambda}\left(e^{-\lambda x}-e^{-\mu x}\right)$$

**1.8** Let $X_1,X_2$ and $X_3$ be i.i.d random variables with distribution $\text{Uni}(0,1)$. Find the density functions of the random variables $Y:=X_1+X_2$ and $Z:=X_1+X_2+X_3$.

**Solution** The density of a random variable with $\text{Uni}(0,1)$ distribution is $f(x)=\mathbb{1}_{[0,1]}$. Let $g(x)$ denote the density of $Y$. Then

$$g(x)=\int_{-\infty}^{\infty}f(y)f(x-y)\,\mathrm{d}y$$

Clearly for any $x<0$ and $2\le x$ we have $g(x)=0$. Considering the following cases

(i) $0\le x<1$

$$g(x)=\int_0^x 1dy=x$$

(ii) $1 \leq x < 2$

$$g(x) = \int_{x-1}^{1} 1dy = 2 - x$$

We concluded

$$g(x) = x\mathbb{1}_{[0,1)}(x) + (2-x)\mathbb{1}_{[1,2)}(x).$$

Clearly for any $x \leq 0$ and $3 \leq x$ we have $h(x) = 0$. Now let $h(x)$ denote the density of $Z$. Then we have

$$h(x) = \int_{-\infty}^{\infty} g(y)f(x-y)\,\mathrm{d}y$$

Considering the following cases

(i) $0 \leq x < 1$

$$h(x) = \int_{0}^{x} g(y)\,\mathrm{d}y = \int_{0}^{x} y\,\mathrm{d}y = \frac{x^2}{2}$$

(ii) $1 \leq x < 2$

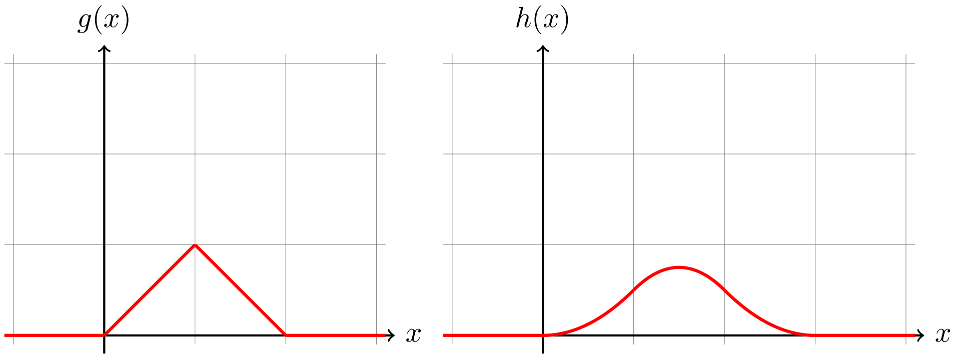
$$h(x) = \int_{x-1}^{x} g(y)\,\mathrm{d}y = \int_{x-1}^{1} g(y)\,\mathrm{d}y + \int_{1}^{x} g(y)\,\mathrm{d}y$$
$$= \int_{x-1}^{1} y\,\mathrm{d}y + \int_{1}^{x} 2-y\,\mathrm{d}y = 3x - x^2 - \frac{3}{2}$$

(iii) $2 \leq x < 3$

$$h(x) = \int_{x-1}^{2} g(y)\,\mathrm{d}y = \int_{x-1}^{2} 2-y\,\mathrm{d}y = \frac{9}{2} - 3x + \frac{x^2}{2}$$

We concluded

$$h(x) = \frac{x^2}{2}\mathbb{1}_{[0,1)}(x) + \left(3x - x^2 - \frac{3}{2}\right)\mathbb{1}_{[1,2)}(x) + \left(\frac{9}{2} - 3x + \frac{x^2}{2}\right)\mathbb{1}_{[2,3)}(x)$$




**1.9** Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d random variables with distribution Uni$(0,1)$. Denote $f_n(x)$ the density function of $S_n := \sum_{k=1}^{n} X_k$. Prove that

$$f_n(x) = \frac{1}{(n-1)!}\sum_{k=0}^{[x]}(-1)^k\binom{n}{k}(x-k)^{n-1}.$$

Using a computer program (e.g. *Mathematica, Python*) plot the graph of the function

$$\widetilde{f}_n(x) := \sqrt{\frac{n}{12}} f_n\left(\frac{n}{2} + \sqrt{\frac{n}{12}}x\right)$$

for $n = 1, 2, \ldots, 10$. What do we see? Interpret the result!

**Solution** We will prove it by induction on $n \in \mathbb{N}^+$. Clearly for $n = 1$ we have

$$f_1(x) = \mathbb{1}_{[0,1)}(x),$$

which is exactly the density of $S_1 = X_1 \sim \text{Uni}(0,1)$.

Now suppose it holds for some $n \in \mathbb{N}^+$. Then

$$f_{n+1}(x) = \int_{-\infty}^{\infty} f_n(x-y)\mathbb{1}_{[0,1)}(y)\,\mathrm{d}y = \int_0^1 f_n(x-y)\,\mathrm{d}y$$

Thus by the induction hypothesis

$$f_{n+1}(x) = \int_0^1 \frac{1}{(n-1)!}\sum_{k=0}^{[x-y]}(-1)^k\binom{n}{k}(x-y-k)^{n-1}\,\mathrm{d}y =$$

$$\underbrace{\int_0^{\{x\}} \frac{1}{(n-1)!}\sum_{k=0}^{[x]}(-1)^k\binom{n}{k}(x-y-k)^{n-1}\,\mathrm{d}y}_{I_1} + \underbrace{\int_{\{x\}}^1 \frac{1}{(n-1)!}\sum_{k=0}^{[x]-1}(-1)^k\binom{n}{k}(x-y-k)^{n-1}\,\mathrm{d}y}_{I_2}.$$

Where

$$\begin{aligned} I_1 =& \frac{1}{(n-1)!}\sum_{k=0}^{[x]}(-1)^k\binom{n}{k}\left[-\frac{(x-y-k)^n}{n!}\right]_{y=0}^{y=\{x\}} \\ =& \frac{1}{n!}\sum_{k=0}^{[x]}(-1)^{k+1}\binom{n}{k}(x-\{x\}-k)^n - \frac{1}{n!}\sum_{k=0}^{[x]}(-1)^{k+1}\binom{n}{k}(x-k)^n \end{aligned}$$

and

$$\begin{aligned} I_2 =& \frac{1}{(n-1)!}\sum_{k=0}^{[x]-1}(-1)^k\binom{n}{k}\left[-\frac{(x-y-k)^n}{n!}\right]_{y=\{x\}}^{1} \\ =& \frac{1}{n!}\sum_{k=0}^{[x]-1}(-1)^{k+1}\binom{n}{k}(x-1-k)^n - \frac{1}{n!}\sum_{k=0}^{[x]-1}(-1)^{k+1}\binom{n}{k}(x-\{x\}-k)^n. \end{aligned}$$

Then

$$\begin{aligned} f_{n+1}(x) =& I_1 + I_2 \\ =& \frac{1}{n!}(-1)^{[x]+1}\underbrace{(x-\{x\}-[x])^n}_{=0} \\ &+ \frac{1}{n!}\sum_{k=0}^{[x]-1}(-1)^{k+1}\binom{n}{k}(x-1-k)^n - \frac{1}{n!}\sum_{k=0}^{[x]}(-1)^{k+1}\binom{n}{k}(x-k)^n \\ =& \frac{1}{n!}\sum_{k=1}^{[x]}(-1)^k\binom{n}{k-1}(x-k)^n + \frac{1}{n!}\sum_{k=0}^{[x]}(-1)^k\binom{n}{k}(x-k)^n \\ =& \frac{x^n}{n!} + \frac{1}{n!}\sum_{k=1}^{[x]}(-1)^k\left(\binom{n}{k-1}+\binom{n}{k}\right)(x-k)^n \\ =& \frac{x^n}{n!} + \frac{1}{n!}\sum_{k=1}^{[x]}(-1)^k\binom{n+1}{k}(x-k)^n \\ =& \frac{1}{n!}\sum_{k=0}^{[x]}(-1)^k\binom{n+1}{k}(x-k)^n. \end{aligned}$$

Therefore, the formula also holds for $n+1$.

**1.10** Let $X$ and $Y$ be independent r.v. with dist. $\mathrm{Poi}(\lambda)$ and $\mathrm{Uni}(0,1)$. Find the distribution of $Z := X + Y$.

**Solution** Let $F$ denote the CDF of $X$ and $g$ the density function of $Y$. Then for $z \geq 0$

$$\begin{aligned}
\mathbb{P}(Z \leq z) &= \int_{-\infty}^{\infty} F(z-x) \cdot g(x) \,\mathrm{d}x \\
&= \int_0^1 F(z-x) \,\mathrm{d}x \\
&= \int_0^1 \mathbb{P}(X \leq z) \,\mathrm{d}x \\
&= \int_0^1 \sum_{k=0}^{\lfloor z-x \rfloor} e^{-\lambda} \frac{\lambda^k}{k!} \,\mathrm{d}x \\
&= \int_0^{\{z\}} \sum_{k=0}^{\lfloor z-x \rfloor} e^{-\lambda} \frac{\lambda^k}{k!} \,\mathrm{d}x + \int_{\{z\}}^1 \sum_{k=0}^{\lfloor z-x \rfloor} e^{-\lambda} \frac{\lambda^k}{k!} \,\mathrm{d}x \\
&= \int_0^{\{z\}} \sum_{k=0}^{\lfloor z \rfloor} e^{-\lambda} \frac{\lambda^k}{k!} \,\mathrm{d}x + \int_{\{z\}}^1 \sum_{k=0}^{\lfloor z \rfloor - 1} e^{-\lambda} \frac{\lambda^k}{k!} \,\mathrm{d}x \\
&= \{z\} \cdot \sum_{k=0}^{\lfloor z \rfloor} e^{-\lambda} \frac{\lambda^k}{k!} + (1 - \{z\}) \cdot \sum_{k=0}^{\lfloor z \rfloor - 1} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \{z\} \cdot e^{-\lambda} \frac{\lambda^{\lfloor z \rfloor}}{\lfloor z \rfloor !} + \sum_{k=0}^{\lfloor z \rfloor - 1} e^{-\lambda} \frac{\lambda^k}{k!}.
\end{aligned}$$