2nd Exercise Sheet
Convolutions II

2.1 We say that a random variable X has Cauchy distribution with parameters $m \in \mathbb{R}$ and $\tau > 0$ (notation: $\text{Cau}(m, \tau)$) if its density function is

$$f(x) = \frac{\tau}{\pi(\tau^2 + (x-m)^2)} \qquad \text{for } x \in \mathbb{R}.$$

    (a) Show that if X has distribution $\text{Cau}(0,1)$ then for real numbers $\tau, m \in \mathbb{R}$, $\tau X + m$ has distribution $\text{Cau}(m, |\tau|)$.

    (b) Let X and Y by absolutely continuous random variables with density functions f and g respectively. If the density of $Z = X + Y$ is $f * g$, does this imply that X and Y are independent?

HW 2.2 Show that it is possible to find example like **2.1**b for discrete distributions. Let X and Y be random variables taking values in $\{0, 1, 2\}$ such that $\mathbb{P}(X = i) = \mathbb{P}(Y = i) = 1/3$ for every $i = 0, 1, 2$.

    (a) Find the distribution of $X + Y$ assuming that X and Y are independent.

    (b) Find all joint distributions (X, Y) so that the distribution of $X + Y$ is the same as the answer as **2.2**a.

HW 2.3 Let $X, Y > 0$ be positive and independent random variables with distribution functions F and G respectively. Give the distribution of XY!

2.4 Let X and Y be independent random variables such that $X \sim \text{Uni}(0,1)$ and Y has k-times continuously differentiable distribution function $F(y) = \mathbb{P}(Y < y)$, where $k \in \{0, 1, \ldots\}$. Show that the distribution of $X + Y$ is $(k+1)$-times continuously differentiable.

2.5 Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d random variables with common distribution $\mathbb{P}(X_i = 0) = \frac{1}{2} = \mathbb{P}(X_i = 1)$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. (The sum is convergent with probability 1!) Prove that the distribution of Y is uniform on the interval $[0, 1]$.

HW* 2.6 Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d random variables with common distribution $\text{Uni}(0,1)$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. (The sum is convergent with probability 1!) Prove that the distribution function $F(y) := \mathbb{P}(Y < y)$ of Y is continuous. Moreover, show that F is infinitely differentiable but nowhere analytic. (*Hint:* For the last part show that the radius of convergence of the Taylor series is zero for every point in $[0,1]$.)

2.7 For a given $\lambda, \alpha > 0$, denote $\text{Gamma}(\alpha, \lambda)$ the distribution, of which density function is

$$f_{\alpha, \lambda}(x) := \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \mathbb{1}_{\{x > 0\}}.$$

Calculate the value of $\text{B}(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} \, dx$ for every fixed parameters $a, b > 0$. (*Hint:* Use the definition of $\Gamma(a)\Gamma(b)$ and the substitution $x = zt, y = z(1-t)$ to calculate the double integral $\int_0^\infty \int_0^\infty x^{a-1} e^{-x} y^{b-1} e^{-y} \, dx \, dy$ in two ways!)

2.8 Calculate the moments of the distribution $\text{Gamma}(\alpha, \lambda)$.

HW 2.9 For every $a, b > 0$, denote $\text{Beta}(a, b)$ the distribution, of which density function is

$$f_{a,b}(x) := \frac{1}{\text{B}(a, b)} x^{a-1}(1-x)^{b-1} \mathbb{1}_{\{x \in [0,1]\}}.$$

Calculate the moments of $\text{Beta}(a, b)$! (The constant $\text{B}(a, b)$ is defined in **2.7**.)

2.10 Prove that if X and Y are i.i.d. standard normal random variables, and a and b real numbers then $U = aX + bY$ és $V = bX - aY$ are also independent. What distribution do U and V have?