

2nd Exercise Class

Convolutions II

2.1 We say that a random variable X has Cauchy distribution with parameters $m \in \mathbb{R}$ and $\tau > 0$ (notation: $\text{Cau}(m, \tau)$) if its density function is

$$f(x) = \frac{\tau}{\pi(\tau^2 + (x - m)^2)} \quad \text{for } x \in \mathbb{R}.$$

- (a) Show that if X has distribution $\text{Cau}(0, 1)$ then for real numbers $\tau, m \in \mathbb{R}$, $\tau X + m$ has distribution $\text{Cau}(m, |\tau|)$.
- (b) Let X and Y be absolutely continuous random variables with density functions f and g respectively. If the density of $Z = X + Y$ is $f * g$, does this imply that X and Y are independent?

Solution

(a) Let us find the PDF of $\tau X + m$

(i) If $\tau > 0$:

$$\begin{aligned} \mathbb{P}(\tau X + m < x) &= \mathbb{P}\left(X < \frac{x - m}{\tau}\right) \\ \frac{d}{dx} \mathbb{P}(\tau X + m < x) &= \frac{1}{\tau} \frac{1}{\pi \left(1 + \left(\frac{x - m}{\tau}\right)^2\right)} = \frac{\tau}{\pi (\tau^2 + (x - m)^2)} \end{aligned}$$

(ii) If $\tau < 0$:

$$\begin{aligned} \mathbb{P}(\tau X + m < x) &= \mathbb{P}\left(X > \frac{x - m}{\tau}\right) = 1 - \mathbb{P}\left(X < \frac{x - m}{\tau}\right) \\ \frac{d}{dx} \mathbb{P}(\tau X + m < x) &= -\frac{1}{\tau} \frac{1}{\pi \left(1 + \left(\frac{x - m}{\tau}\right)^2\right)} = \frac{-\tau}{\pi (\tau^2 + (x - m)^2)} \end{aligned}$$

In both cases $\tau X + m$ has the density function

$$x \mapsto \frac{|\tau|}{\pi (|\tau|^2 + (x - m)^2)}.$$

Remark: For $\tau = 0$ we can interpret $\text{Cau}(m, 0)$ as the almost surely constant m variable. Therefore, formally the identity also holds in that case.

- (b) It does not imply. Let $X \sim \text{Cau}(0, 1)$ and $Y = X$. Then X and Y clearly not independent. However, we will show that the density of $Z = X + Y$ is exactly $f * g$. First since $Z = X + Y = X + X = 2X \sim \text{Cau}(0, 2)$, the density of Z is

$$\frac{2}{\pi (4 + x^2)}.$$

Now let us calculate $f * g$.

$$(f * g)(z) = (f * f)(x) = \int_{-\infty}^{\infty} \frac{1}{\pi (1 + (x - z)^2)} \cdot \frac{1}{\pi (1 + z^2)} dz$$

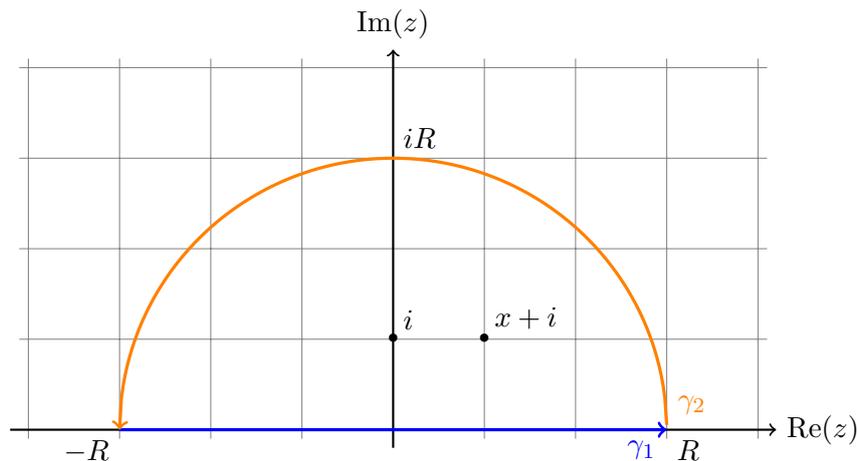
Then notice

$$h(z) = \frac{1}{\pi^2} \frac{1}{(z + i)(z - i)(z - (x - i))(z - (x + i))}.$$

Let us define the function $h : \mathbb{C} \rightarrow \mathbb{C}$ for any $x \in \mathbb{R}$ fixed

$$h(z) := \frac{1}{\pi^2} \cdot \frac{1}{(1 + (x - z)^2)(1 + z^2)}.$$

Let us define the following curves for $R > 0$.



Then let $\Gamma = \gamma_1 \cup \gamma_2$. First, since $|h(z)| = R$ for all $z \in \gamma_2$ and the dominant term in the denominator is z^4 as $R \rightarrow \infty$

$$\left| \int_{\gamma_2} h(z) dz \right| \leq L(\gamma_2) \max_{z \in \gamma_2} |h(z)| = \pi R \cdot \max_{z \in \gamma_2} |h(z)| \sim \pi R \cdot \frac{1}{R^4} = \frac{\pi}{R^3} \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Therefore,

$$\lim_{R \rightarrow \infty} \int_{\Gamma} h(z) dz = \lim_{R \rightarrow \infty} \int_{\gamma_1} h(z) dz = (f * f)(x).$$

Clearly, h is complex differentiable on \mathbb{C} except the points $i, -i, x+i$ and $x-i$. Suppose $x \neq 0$. In this case $i \neq x+i$ and due to the Residue theorem for large enough $R > 0$ (e.g. $R > 1 + |x|$)

$$\int_{\Gamma} h(z) dz = 2\pi i (\text{Res}(h, i) + \text{Res}(h, x+i)).$$

The residues are (of order one since $i \neq x+i$)

$$\text{Res}(h, i) = \lim_{z \rightarrow i} (z - i) \cdot h(z) = \frac{1}{\pi^2 (1 + (x - i)^2) 2i}$$

and

$$\text{Res}(h, x+i) = \lim_{z \rightarrow x+i} (z - (x+i)) \cdot h(z) = \frac{1}{\pi^2 (1 + (x+i)^2) 2i}.$$

It follows that

$$(f * f)(x) = 2\pi i \left(\frac{1}{\pi^2 (1 + (x - i)^2) 2i} + \frac{1}{\pi^2 (1 + (x + i)^2) 2i} \right) = \frac{2}{\pi(4 + x^2)}.$$

This already determines the density for $x \neq 0$. Since changing the density at a single point does not change the distribution, we are done.

However, it can be shown that for $x = 0$ the identity also holds. It is true since

$$\int_{\Gamma} h(z) dz = 2\pi i \cdot \text{Res}(h, i).$$

Now we have a residue of second order. Therefore,

$$\text{Res}(h, i) = \lim_{z \rightarrow i} \frac{d}{dz} (z - i)^2 \cdot h(z) = \frac{1}{4\pi^2 i}.$$

Thus,

$$(f * f)(0) = 2\pi i \cdot \frac{1}{4\pi^2 i} = \frac{1}{2\pi} = \frac{2}{\pi(4 + 0^2)}.$$

2.5 Let $X_1, X_2, \dots, X_n, \dots$ be i.i.d random variables with common distribution $\mathbb{P}(X_i = 0) = \frac{1}{2} = \mathbb{P}(X_i = 1)$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. (The sum is convergent with probability 1!) Prove that the distribution of Y is uniform on the interval $[0, 1]$.

Solution Notice that $\mathbb{P}(0 \leq Y \leq 1) = 1$ and

$$Y = \sum_{n=1}^{\infty} \frac{X_n}{2^n} = \frac{1}{2} X_1 + \underbrace{\frac{1}{2} \sum_{n=1}^{\infty} \frac{X_{n+1}}{2^n}}_{:=Y_1 \stackrel{d}{=} Y}.$$

Then we have

$$F(z) := \mathbb{P}(Y < z) = \mathbb{P}\left(\frac{X_1}{2} + \frac{Y_1}{2} < z\right) = \mathbb{P}(X_1 + Y_1 < 2z).$$

$$G(x) := \mathbb{P}(X_1 < x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{2} & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 < x \end{cases}$$

Since $Y_1 \stackrel{d}{=} Y$, we have $F(z) = \mathbb{P}(Y_1 < z)$ and

$$F(z) = \int_{-\infty}^{\infty} F(2z - x) dG(x) = \frac{1}{2} F(2z) + \frac{1}{2} F(2z - 1) \tag{1}$$

We show that the function

$$\tilde{F}(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \end{cases}$$

satisfies equation (1). Clearly, it is enough to show it for $x \in [0, 1]$.

(i) If $0 \leq x \leq \frac{1}{2}$

$$\frac{1}{2} \tilde{F}(2x) + \frac{1}{2} \underbrace{\tilde{F}(2x - 1)}_{=0} = \frac{1}{2} \tilde{F}(2x) = \frac{1}{2} \cdot 2x = x = \tilde{F}(x).$$

(ii) If $\frac{1}{2} \leq x \leq 1$

$$\frac{1}{2} \underbrace{\tilde{F}(2x)}_{=1} + \frac{1}{2} \tilde{F}(2x - 1) = \frac{1}{2} + \frac{1}{2}(2x - 1) = x = \tilde{F}(x).$$

Moreover, let F_1 and F_2 be two functions satisfying equation (1). Then notice

$$\sup_{x \in [0, \frac{1}{2}]} |F_1(x) - F_2(x)| = \sup_{x \in [0, \frac{1}{2}]} \left| \frac{1}{2} F_1(2x) - \frac{1}{2} F_2(2x) \right| = \frac{1}{2} \sup_{x \in [0, 1]} |F_1(x) - F_2(x)|$$

and

$$\sup_{x \in [\frac{1}{2}, 1]} |F_1(x) - F_2(x)| = \sup_{x \in [\frac{1}{2}, 1]} \left| \frac{1}{2} F_1(2x - 1) - \frac{1}{2} F_2(2x - 1) \right| = \frac{1}{2} \sup_{x \in [0, 1]} |F_1(x) - F_2(x)|.$$

Therefore,

$$\sup_{x \in [0, 1]} |F_1(x) - F_2(x)| = \max \left\{ \sup_{x \in [0, \frac{1}{2}]} |F_1(x) - F_2(x)|, \sup_{x \in [\frac{1}{2}, 1]} |F_1(x) - F_2(x)| \right\} = \frac{1}{2} \sup_{x \in [0, 1]} |F_1(x) - F_2(x)|.$$

It follows that

$$\sup_{x \in [0, 1]} |F_1(x) - F_2(x)| = 0.$$

Thus, the only solution is \tilde{F} . It follows $Y \sim \text{Uni}(0, 1)$.

2.7 For a given $\lambda, \alpha > 0$, denote $\text{Gamma}(\alpha, \lambda)$ the distribution, of which density function is

$$f_{\alpha, \lambda}(x) := \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \mathbb{1}_{\{x>0\}}.$$

Calculate the value of $B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$ for every fixed parameters $a, b > 0$. (*Hint: Use the definition of $\Gamma(a)\Gamma(b)$ and the substitution $x = zt, y = z(1-t)$ to calculate the double integral $\int_0^\infty \int_0^\infty x^{a-1} e^{-x} y^{b-1} e^{-y} dx dy$ in two ways!*)

Solution Consider the integral

$$I = \int_0^\infty \int_0^\infty x^{a-1} e^{-x} y^{b-1} e^{-y} dy dx.$$

(1) Using the product structure

$$\begin{aligned} I &= \int_0^\infty x^{a-1} e^{-x} dx \int_0^\infty y^{b-1} e^{-y} dy \\ &= \Gamma(a)\Gamma(b). \end{aligned}$$

(2) Using the substitution $x = zt$ and $y = z(1-t)$ the boundaries are

$$z = x + y \in [0, \infty), \quad t = \frac{x}{x + y} \in [0, 1].$$

The Jacobian in this case

$$\begin{vmatrix} \frac{dx}{dz} & \frac{dx}{dt} \\ \frac{dy}{dz} & \frac{dy}{dt} \end{vmatrix} = \begin{vmatrix} t & z \\ 1-t & -z \end{vmatrix} = -z.$$

It follows

$$\begin{aligned} I &= \int_0^\infty \int_0^1 (zt)^{a-1} e^{-zt} (z(1-t))^{b-1} e^{-z(1-t)} | -z | dt dz \\ &= \int_0^\infty \int_0^1 z^{(a+b)-1} e^{-z} t^{a-1} (1-t)^{b-1} dt dz \\ &= \int_0^\infty z^{(a+b)-1} e^{-z} dz \int_0^1 t^{a-1} (1-t)^{b-1} dt \\ &= \Gamma(a+b)B(a, b). \end{aligned}$$

Comparing (1) and (2)

$$\begin{aligned} \Gamma(a+b)B(a, b) &= \Gamma(a)\Gamma(b) \\ B(a, b) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \end{aligned}$$

2.8 Calculate the moments of the distribution $\text{Gamma}(\alpha, \lambda)$.

Solution Let $X \sim \text{Gamma}(\alpha, \lambda)$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} \mathbb{E}(X^n) &= \int_0^\infty x^n \cdot \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x} dx = \int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{(\alpha+n)-1} e^{-\lambda x} dx \\ &= \frac{1}{\lambda^n} \int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^{\alpha+n} x^{(\alpha+n)-1} e^{-\lambda x} dx = \frac{1}{\lambda^n} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \underbrace{\int_0^\infty \frac{1}{\Gamma(\alpha+n)} \lambda^{\alpha+n} x^{(\alpha+n)-1} e^{-\lambda x} dx}_{=1}, \end{aligned}$$

where we have the density of $\text{Gamma}(\alpha + n, \lambda)$ integrated on $[0, \infty)$. Therefore,

$$\mathbb{E}(X^n) = \frac{1}{\lambda^n} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$