3rd Exercise Sheet
Generating functions I

3.1 Determine the probability generating function $P(z) = \mathbb{E}\left(z^X\right)$ of $\mathrm{Bin}(n,p)$, $\mathrm{Poi}(\lambda)$ and $\mathrm{Geo}(p)$.

3.2 Let X be an $\mathbb{N}$ valued random variable and denote $P(z)$ the prob. generating function of X. Give the prob. gen. function of $Y := X + k$ and $Z := nX$ $(k, n \in \mathbb{N})$.

3.3 Can the following functions be a prob. generating function of some distribution?

$$(a) \quad \exp\left(\frac{z-1}{\lambda}\right), \quad \lambda > 0; \qquad (b) \quad \frac{(z+1)^6}{64}; \quad (c) \quad \frac{2}{2-z}; \qquad (d) \quad \frac{2}{1+z}.$$

3.4 Let U be a random variable with distribution $\mathrm{Uni}(0,1)$, and let X be the random variable, which conditional distribution is $\mathrm{Bin}(n,U)$ conditioned on U. Determine the distribution of X.

HW 3.5 Let Λ be a random variable with distribution $\mathrm{Exp}(\mu)$ and let X be the random variable, which conditional distribution is $\mathrm{Poi}(\Lambda)$ conditioned on Λ. Determine the distribution of X.

3.6 Let X be an $\mathbb{N}$ valued random variable and denote $P(z)$ the prob. generating function of X. Give the generating functions of the following sequences: $a_n := \mathbb{P}(X \le n)$, $b_n := \mathbb{P}(X < n)$, $c_n := \mathbb{P}(X \ge n)$, $d_n := \mathbb{P}(X > n+1)$ and $e_n := \mathbb{P}(X = 2n)$. (Note that these are not prob. distributions.)

3.7 Let X and Y be independent $\mathbb{N}$-valued random variables with prob. gen. function $U(z)$ and $V(z)$ respectively. Prove that the coefficients of z^j in the Laurent series of the function $K(z) := U(z)V(1/z)$ are $b_j := \mathbb{P}(X - Y = j)$, for $j \in \mathbb{Z}$.

3.8 Let $\xi_1, \xi_2, \ldots$ be i.i.d random variables with distribution: $\mathbb{P}(\xi_i = 1) = p$, $\mathbb{P}(\xi_i = 0) = 1 - p$, where $0 < p < 1$. Let

$$\nu_{\alpha\beta} := \min\{n \ge 2 : \xi_{n-1} = \alpha,\ \xi_n = \beta\}, \qquad \alpha, \beta \in \{0,1\}.$$

Determine the prob. gen. function of $\nu_{\alpha\beta}$, and using this, the expected value and variance, for every possible combination of α, β.

HW 3.9 Let $\xi_1, \xi_2, \ldots$ be i.i.d. $\mathbb{N}$-valued random variables such that $\mathbb{P}(\xi_j = n) = p_n$ for $n = 0, 1, 2, \ldots$. Let $\nu = \min\{k \ge 2 : \xi_{k-1} = \xi_k\}$. Find the probability generating function and the expected value of ν!

(Hint: express $\mathbb{E}(z^\nu | \xi_1 = n)$ by using the second toss with $\mathbb{E}(z^\nu)$.)

3.10 We call the distribution of a random variable X *infinitely divisible* for every $n \ge 1$ there exist i.i.d random variables $Y_{1,n}, \ldots, Y_{n,n}$ such that $\sum_{k=1}^n Y_{k,n}$ and X have the same distribution. Are the binomial, Poission and geometric distributions infinitely divisible?

3.11 Let $X_1, X_2, \ldots$ be i.i.d. $\mathbb{N}$-valued random variables and let ν be an $\mathbb{N}$-valued random variable independent of X_i's. Let $Y = \sum_{k=1}^\nu X_k$. Show that

$$\mathbb{E}(Y) = \mathbb{E}(\nu)\mathbb{E}(X_1) \qquad \text{and} \qquad \mathbb{D}^2(Y) = \mathbb{D}^2(\nu)\mathbb{E}(X_1)^2 + \mathbb{E}(\nu)\mathbb{D}^2(X_1).$$

HW 3.12 Let $X_1, X_2, \ldots$ be i.i.d random variables with distribution (optimistic) $\mathrm{Geo}(p_1)$ $(\mathbb{P}(X_i = k) = p_1(1-p_1)^{k-1},\ k = 1, 2, \ldots)$, and let ν be a random variable independent of X_i's and with distribution (optimistic) $\mathrm{Geo}(p_2)$ $(\mathbb{P}(\nu = k) = p_2(1-p_2)^{k-1},\ k = 1, 2, \ldots)$. Show by using prob. gen. functions that

$$\sum_{i=1}^{\nu} X_i \text{ has distribution (optimistic) } \mathrm{Geo}(p_1 p_2)!$$

HW* 3.13 Show that for every $0 < p < 1$ there exists a prob. sequence $p_0, p_1, \ldots$ (i.e. $p_k \ge 0$ and $\sum_{k=0}^\infty p_k = 1$) and a parameter $\lambda > 0$, such that $\sum_{i=1}^\nu X_i$ has distribution $\mathrm{Geo}(p)$, where $X_1, X_2, \ldots$ are i.i.d random variables with distribution $\mathbb{P}(X_i = k) = p_k$ for every $k \ge 1$, and ν is independent of X_i with distribution $\mathrm{Poi}(\lambda)$.