3rd Exercise Class
Generating functions I

3.2 Let X be an $\mathbb{N}$ valued random variable and denote $P(z)$ the prob. generating function of X. Give the prob. gen. function of $Y := X + k$ and $Z := nX$ $(k, n \in \mathbb{N})$.

Solution

(1) The PGF of Y is

$$\mathbb{E}\Big(z^Y\Big) = \mathbb{E}\Big(z^{X+k}\Big) = \mathbb{E}\Big(z^k z^X\Big) = z^k \mathbb{E}\Big(z^X\Big) = z^k P(z).$$

(2) The PGF of Z is

$$\mathbb{E}\Big(z^Z\Big) = \mathbb{E}\Big(z^{nX}\Big) = \mathbb{E}\Big((z^n)^X\Big) = P(z^n).$$

3.3 Can the following functions be a prob. generating function of some distribution?

$$(a) \quad \exp\left(\frac{z-1}{\lambda}\right), \quad \lambda > 0; \qquad (b) \quad \frac{(z+1)^6}{64}; \quad (c) \quad \frac{2}{2-z}; \qquad (d) \quad \frac{2}{1+z}.$$

Solution

(a) Notice that

$$\exp\left(\frac{z-1}{\lambda}\right) = \exp\left(\frac{1}{\lambda}(z-1)\right).$$

Therefore, it is the PGF of $\mathrm{Poi}(1/\lambda)$.

(b) Notice that

$$\frac{(z+1)^6}{64} = \left(\frac{z+1}{2}\right)^6 = \left(\frac{1}{2} + \frac{1}{2}z\right)^6 = \left(\left(1 - \frac{1}{2}\right) + \frac{1}{2}z\right)^6.$$

Therefore, it is the PGF of $\mathrm{Bin}(6, 1/2)$.

(c) It cannot be a PGF since

$$\left.\frac{2}{2-z}\right|_{z=1} = 2 > 1.$$

We know that for a PGF P of X

$$P(1) = \mathbb{P}(X < \infty) \leq 1.$$

(d) It cannot be a PFG since it is strictly monotone decreasing on [0,1]

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{2}{1+z} = \frac{-2}{(1+z)^2} < 0.$$

We know that a PGF is monotone increasing on $[0, 1]$.

3.4 Let U be a random variable with distribution $\mathrm{Uni}(0, 1)$, and let X be the random variable, which conditional distribution is $\mathrm{Bin}(n, U)$ conditioned on U. Determine the distribution of X.

Solution Let us determine the PGF P of X. Using the tower property

$$P(z) = \mathbb{E}\Big(z^X\Big) = \mathbb{E}\left(\underbrace{\mathbb{E}\Big(z^X \,|\, U\Big)}_{\text{PGF of } \mathrm{Bin}(n,U)}\right) = \mathbb{E}(((1-U) + Uz)^n).$$

Then

$$P(z) = \int_0^1 ((1-u) + uz)^n \,\mathrm{d}u = \left[\frac{(1 + (z-1)u)^{n+1}}{(n+1)(z-1)}\right]_{u=0}^{u=1} = \frac{1}{n+1}\frac{z^{n+1} - 1}{z-1}.$$

Using the identity

$$(z-1)\sum_{k=0}^{n} z^k = z^{n+1} - 1$$

we obtain

$$P(z) = \sum_{k=0}^{n} \frac{1}{n+1} z^k.$$

It follows that

$$\mathbb{P}(X=k) = \begin{cases} \frac{1}{n+1} & \text{if } k = 0, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

We conclude $X \sim \text{Uni}\{0, 1, \dots, n\}$.

3.6 Let X be an $\mathbb{N}$ valued random variable and denote $P(z)$ the prob. generating function of X. Give the generating functions of the following sequences: $a_n := \mathbb{P}(X \le n)$, $b_n := \mathbb{P}(X < n)$, $c_n := \mathbb{P}(X \ge n)$, $d_n := \mathbb{P}(X > n+1)$ and $e_n := \mathbb{P}(X = 2n)$. (Note that these are not prob. distributions.)

Solution The PGF of the sequence a_n for $z \in [0, 1)$ is

$$\begin{aligned} A(z) &= \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \mathbb{P}(X \le n) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathbb{P}(X=k) z^n = \sum_{k=0}^{\infty} \sum_{k=n}^{\infty} \mathbb{P}(X=k) z^n \\ &= \sum_{k=0}^{\infty} \mathbb{P}(X=k) \sum_{n=k}^{\infty} z^n = \sum_{k=0}^{\infty} \mathbb{P}(X=k) \frac{z^k}{1-z} = \frac{1}{1-z} \sum_{k=0}^{\infty} \mathbb{P}(X=k) z^k = \frac{1}{1-z} P(z). \end{aligned}$$

3.8 Let $\xi_1, \xi_2, \dots$ be i.i.d random variables with distribution: $\mathbb{P}(\xi_i = 1) = p$, $\mathbb{P}(\xi_i = 0) = 1 - p$, where $0 < p < 1$. Let

$$\nu_{\alpha\beta} := \min\{n \ge 2 : \xi_{n-1} = \alpha,\ \xi_n = \beta\}, \qquad \alpha, \beta \in \{0, 1\}.$$

Determine the prob. gen. function of $\nu_{\alpha\beta}$, and using this, the expected value and variance, for every possible combination of α, β.

Solution We solve it for $\alpha = 0$ and $\beta = 1$. In this case the PGF of $\nu = \nu_{01}$ is after conditioning on ξ_1 is

$$\begin{aligned} P(z) &= \mathbb{E}(z^\nu) \\ &= \mathbb{E}(z^\nu \,|\xi_1 = 0)(1-p) + \mathbb{E}(z^\nu \,|\xi_1 = 1)p \\ &= \mathbb{E}(z^\nu \,|\xi_1 = 0)(1-p) + \mathbb{E}\Big(z^{\nu+1}\Big)p \\ &= \mathbb{E}(z^\nu \,|\xi_1 = 0)(1-p) + pzP(z). \end{aligned}$$

Moreover, notice that

$$\begin{aligned} \mathbb{E}(z^\nu \,|\xi_1 = 0) &= \mathbb{E}(z^\nu \,|\xi_1 = 0, \xi_2 = 0)(1-p) + \mathbb{E}(z^\nu \,|\xi_1 = 0, \xi_2 = 1)p \\ &= \mathbb{E}\Big(z^{\nu+1} \,|\xi_1 = 0\Big)(1-p) + \mathbb{E}\Big(z^2\Big)p \\ &= (1-p)z\mathbb{E}(z^\nu \,|\xi_1 = 0) + pz^2. \end{aligned}$$

It follows that

$$\mathbb{E}(z^\nu \,|\xi_1 = 0) = \frac{pz^2}{1-(1-p)z}.$$

Substituting this back into the first equation

$$P(z) = \frac{pz^2}{1-(1-p)z}(1-p) + pzP(z).$$

Therefore, the PGF of ν is

$$P(z) = \frac{p(1-p)z^2}{(1-pz)(1-(1-p)z)}.$$

3.11 Let $X_1, X_2, \ldots$ be i.i.d. $\mathbb{N}$-valued random variables and let ν be an $\mathbb{N}$-valued random variable independent of X_i's. Let $Y = \sum_{k=1}^{\nu} X_k$. Show that

$$\mathbb{E}(Y) = \mathbb{E}(\nu)\mathbb{E}(X_1) \qquad \text{and} \qquad \mathbb{D}^2(Y) = \mathbb{D}^2(\nu)\mathbb{E}(X_1)^2 + \mathbb{E}(\nu)\mathbb{D}^2(X_1).$$

Solution Let P denote the PGF of X_i's and G the PGF of ν. We know that the PGF of Y in this case is $H = G \circ P$. The expected value of Y can be expressed as

$$\mathbb{E}(Y) = H'(1).$$

More precisely, $\mathbb{E}(Y) = \lim_{z \nearrow 1} H'(z)$ but we use the previous short notation for simplicity. Therefore,

$$\mathbb{E}(Y) = H'(1) = \frac{\mathrm{d}}{\mathrm{d}z} G(P(z))\bigg|_{z=1} = G'(P(1))P'(1) = G'(1)P'(1) = \mathbb{E}(X_1)\mathbb{E}(\nu)$$

since we can assume that the variables are almost surely finite (thus $P(1) = 1$).