4th Exercise Sheet

Generating functions II

4.1 Let $G(s,t)$ be the joint probability generating function of the random variables X and Y. That is $G(s,t) = \mathbb{E}\left(s^X t^Y\right)$. Prove that $G(s,1)$ is the prob. gen. function of X, and $G(1,t)$ is the prob. gen. function of Y. Moreover, show that

$$\mathbb{E}(XY) = \frac{\partial^2}{\partial s \partial t} G(s,t)\Big|_{s=t=1}.$$

What does the relation $G(s,t) = G(s,1) \cdot G(1,t)$ mean? What is $G(s,s)$?

HW 4.2 Prove that the function

$$G(x,y,z,w) = \frac{1}{8}\left(xyzw + xy + yz + zw + zx + yw + xw + 1\right)$$

is the joint prob. generating function of 4 random variables such that any two and any three out of them are independent, but all the four are not independent.

HW 4.3 Let us consider a branching process for which the expected value of the offsprings is ν and the variance is σ^2. Denote X_n the number of individuals in the nth generation. That is, $X_0 = 1$, X_1 is the number of children of the first individual, etc. Find and prove the formula for the expectation and the variance of the nth generation!

(a) Give a formula for $\mathbb{E}(X_n)$!
(b) Give a formula for $\mathbb{D}^2(X_n)$ in case $\nu \neq 1$!
(c) Give a formula for $\mathbb{D}^2(X_n)$ in case $\nu = 1$!

4.4 Denote $\theta(p)$ the probability that a branching process with offspring distribution $\mathrm{Geo}(p)$ never extinct. Plot the graph of $p \mapsto \theta(p)$!

4.5 Let us consider a branching process, for which the probability generating function of the offsprings is $P(z)$. Denote X the size of the whole population (i.e. the number of all individuals who ever lived). Denote $Q(z) = \mathbb{E}\left(z^X\right)$. Prove that $Q(z)$ is the inverse of $z/P(z)$!

4.6 (a) What is the probability that a branching process with successor distribution $\mathrm{Geo}(1/2)$ survives until the nth generation? (Here, Y has distribution $\mathrm{Geo}(1/2)$ if $\mathbb{P}(Y = k) = (1/2)^{k+1}$ for every $k \geq 0$.)
(b) Denote X the size of the complete population. That is, if Y_n denotes the cardinality of the nth generation then $X = \sum_{n=0}^{\infty} Y_n$. What is

$$\lim_{k\to\infty} k^{3/2}\mathbb{P}(X = k) = ?$$

4.7 Let $\xi_1, \xi_2, \ldots$ be i.i.d random variables such that $\mathbb{P}(\xi_i = \pm 1) = \frac{1}{2}$. Denote $S_n = \sum_{i=1}^{n} \xi_i$ the simple random walk on $\mathbb{Z}$. Let $\tau = \min\{n \geq 1 : S_n = 1\}$. Calculate $\mathbb{P}(\tau = k)$! What is the limit

$$\lim_{k\to\infty} k^{3/2}\mathbb{P}(\tau = 2k-1) = ?$$

HW 4.8 An amoebae can do during a day only two things: it splits into two with probability $\frac{1}{2}$, and dies with prob. $\frac{1}{2}$. Denote X the number of the branches of the branching process. Show that X and τ have the same distribution, where τ is defined in **4.7**. What is the meaning of this?

4.9 Let $\xi_1, \xi_2, \ldots$ be i.i.d random variables such that $\mathbb{P}(\xi_i = 1) = p$ and $\mathbb{P}(\xi_i = -1) = q$ $(p + q = 1)$. Denote $S_n = \sum_{i=1}^{n} \xi_i$ the simple random walk on $\mathbb{Z}$. Calculate the generating function L of the last visit to the origin $\lambda = \sup\{n \geq 0 : S_n = 0\}$! What is the expected value of λ?

HW* 4.10 Consider the (infinite) graph $\mathbb{G}_g$, which is a homogeneous tree with degree g and a symmetric random walk on it. That is, let S_n be a random walk on $\mathbb{G}_g$, which starts at a given vertex (the root) and in every time unit, it chooses one out of the g neighbours with uniform probability g^{-1} and steps there. Determine the functions Φ, F and L. ($\Phi(z)$: the prob. gen. function of the first hitting time of a given neighbour, $F(z)$ the prob. gen. function of the first return to the root; $L(z)$ the prob. gen. function of the last hitting time of the root.)