4th Practice Class
Generating functions II

4.1 Let $G(s,t)$ be the joint probability generating function of the random variables X and Y. That is $G(s,t) = \mathbb{E}\left(s^X t^Y\right)$. Prove that $G(s,1)$ is the prob. gen. function of X, and $G(1,t)$ is the prob. gen. function of Y. Moreover, show that

$$\mathbb{E}(XY) = \frac{\partial^2}{\partial s \partial t} G(s,t)\Big|_{s=t=1}.$$

What does the relation $G(s,t) = G(s,1) \cdot G(1,t)$ mean? What is $G(s,s)$?

Solution We have

$$G(s,1) = \mathbb{E}\left(s^X 1^Y\right) = \mathbb{E}\left(s^X\right),$$
$$G(1,t) = \mathbb{E}\left(1^X t^Y\right) = \mathbb{E}\left(t^Y\right).$$

By the definition

$$G(s,t) = \mathbb{E}\left(s^X t^Y\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(X=n, Y=k) s^n t^k$$

thus

$$\frac{\partial^2}{\partial s \partial t} G(s,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(X=n, Y=k) n s^{n-1} k t^{k-1}.$$

It follows that

$$\frac{\partial^2}{\partial s \partial t} G(s,t)\Big|_{s=t=1} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(X=n, Y=k) nk = \mathbb{E}(XY).$$

The realtion $G(s,t) = G(s,1) \cdot G(1,t)$ means that for any $s,t \in [0,1)$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(X=n, Y=k) s^n t^k$$
$$= \sum_{n=0}^{\infty} \mathbb{P}(X=n) s^n \cdot \sum_{k=0}^{\infty} \mathbb{P}(Y=k) t^k = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(X=n)\mathbb{P}(Y=k) s^n t^k$$

Thus for any $n,k \in \mathbb{N}$ we have

$$\mathbb{P}(X=n, Y=k) = \mathbb{P}(X=n)\mathbb{P}(Y=k).$$

It exactly means that X and Y are independent.

By definition

$$G(s,s) = \mathbb{E}\left(s^X s^Y\right) = \mathbb{E}\left(s^{X+Y}\right).$$

Thus $G(s,s)$ is the probability generating function of $X+Y$.

4.4 Denote $\theta(p)$ the probability that a branching process with offspring distribution $\mathrm{Geo}(p)$ never extinct. Plot the graph of $p \mapsto \theta(p)$!

Solution Let P be the PGF of the offsprings. That is, the PGF of $\mathrm{Geo}(p)$

$$P(z) = \frac{p}{1-(1-p)z}.$$

Then we know that

$$\begin{aligned}1-\theta(p) &= \mathbb{P}(\text{the branching process dies out in finite time})\\ &= \inf\left\{z \in [0,1] : P(z) = z\right\}.\end{aligned}$$

Then

$$\begin{aligned} z &= P(z) \\ z &= \frac{p}{1-(1-p)z} \\ 0 &= (z-1)((1-p)z-p). \end{aligned}$$

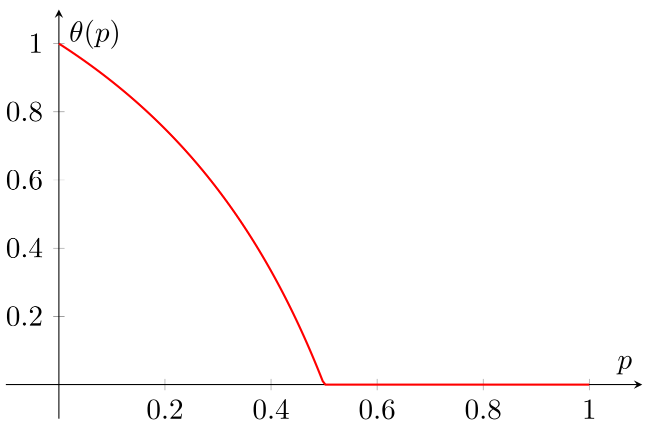
The fixed points are

$$z_1 = 1 \quad \text{and} \quad z_2 = \frac{p}{1-p}.$$

If $p > \frac{1}{2}$ then $z_2 < z_1$, otherwise $z_1 \leq z_2$. Then

$$1-\theta(p) = \begin{cases} \frac{p}{1-p} & \text{if } 0 \leq p < 1/2, \\ 1 & \text{if } 1/2 \leq p \leq 1. \end{cases}$$

Thus the function $p \mapsto \theta(p)$ is



$$p \mapsto \theta(p) = \begin{cases} \frac{1-2p}{1-p} & \text{if } 0 \leq p < 1/2, \\ 0 & \text{if } 1/2 \leq p \leq 1. \end{cases}$$

4.5 Let us consider a branching process, for which the probability generating function of the offsprings is $P(z)$. Denote X the size of the whole population (i.e. the number of all individuals who ever lived). Denote $Q(z) = \mathbb{E}\left(z^X\right)$. Prove that $Q(z)$ is the inverse of $z/P(z)$!

Solution Let X_n be the size of the nth generation. Then by definition $X = \sum_n^\infty X_n$. The PGF of X is

$$\begin{aligned} G(z) = \mathbb{E}\left(z^X\right) = \mathbb{E}\left(z^{\sum_{n=1}^\infty}\right) &= \mathbb{E}\left(z^{1+\sum_{n=2}^\infty X_n}\right) \\ &= z\mathbb{E}\left(z^{\sum_{n=2}^\infty X_n}\right) = z\sum_{k=0}^\infty \mathbb{E}\left(z^{\sum_{n=2}^\infty X_n} \,\middle|\, X_2 = k\right)\mathbb{P}(X_2 = k). \end{aligned}$$

Notice that $\sum_{n=2}^\infty X_k | \, k = 2 \overset{d}{=} \tilde{X}_1 + \cdots + \tilde{X}_k$, where $\tilde{X}_1, \ldots, \tilde{X}_k \overset{d}{=} X$ are i.i.d. since we start k independent branching processes from the descendants of the first individual with the same distribution. Therefore,

$$\begin{aligned} Q(z) &= z\sum_{k=0}^\infty \mathbb{E}\left(z^{\tilde{X}_1+\cdots+\tilde{X}_k}\right)\mathbb{P}(X_2 = k) \\ &= z\sum_{k=0}^\infty \mathbb{E}\left(z^{\tilde{X}_1}\right)\cdots\mathbb{E}\left(z^{\tilde{X}_k}\right)\mathbb{P}(X_2 = k) = z\sum_{K=0}^\infty \mathbb{E}\left(z^X\right)^k\mathbb{P}(X_2 = k). \end{aligned}$$

Since X_2 has the same distribution as the offsprings, we have

$$Q(z) = zP(Q(z)).$$

$Q(z)$ is the inverse of $z/P(z)$ since

$$\frac{Q(z)}{P(Q(z))} = z.$$

4.7 Let $\xi_1, \xi_2, \dots$ be i.i.d random variables such that $\mathbb{P}(\xi_i = \pm 1) = \frac{1}{2}$. Denote $S_n = \sum_{i=1}^n \xi_i$ the simple random walk on $\mathbb{Z}$. Let $\tau = \min\{n \geq 1 : S_n = 1\}$. Calculate $\mathbb{P}(\tau = k)$! What is the limit

$$\lim_{k\to\infty} k^{3/2}\mathbb{P}(\tau = 2k-1) = ?$$

Solution We know that the PGF of τ is

$$P(z) = \frac{1-\sqrt{1-z^2}}{z} = \sum_{k=0}^{\infty} \mathbb{P}(\tau = k) z^k.$$

First let us find the Taylor expansion of $H(z) = 1 - \sqrt{1-z^2}$. The derivatives are

$$H^{(k)}(z) = \frac{1}{2^k} \prod_{i=1}^{k-1} (2i-1)(1-z)^{\frac{2k-1}{2}}.$$

Therefore,

$$H(z) = \sum_{k=1}^{\infty} \frac{H^{(k)}(0)}{k!} z^k = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{1}{2^k} \prod_{i=1}^{k-1}(2i-1) z^k.$$

Moreover,

$$P(z) = \frac{H(z^2)}{z} = \sum_{k=1}^{\infty} \underbrace{\frac{1}{k!}\frac{1}{2^k}\prod_{i=1}^{k-1}(2i-1)}_{\mathbb{P}(\tau=2k-1)} z^{2k-1}.$$

We need to calculate the limit of

$$k^{\frac{3}{2}}\mathbb{P}(\tau = 2k-1) = k^{\frac{3}{2}} \cdot \frac{1}{k!}\frac{1}{2^k}\prod_{i=1}^{k-1}(2i-1)$$

as $k \to \infty$. First, let us find a closed formula for the term $\prod_{i=1}^{k-1}(2i-1)$. Notice that

$$\prod_{i=1}^{k-1}(2i-1) \cdot \prod_{j=1}^{k-1}(2j) = (2(k-1))!$$

$$\prod_{i=1}^{k-1}(2i-1) \cdot 2^{k-1}\prod_{j=1}^{k-1} j = (2(k-1))!$$

$$\prod_{i=1}^{k-1}(2i-1) \cdot 2^{k-1}(k-1)! = (2(k-1))!$$

$$\prod_{i=1}^{k-1}(2i-1) = \frac{(2(k-1))!}{2^{k-1}(k-1)!}.$$

Thus, the limit we want to calculate as $k \to \infty$ is

$$k^{\frac{3}{2}} \frac{1}{k!}\frac{1}{2^k}\frac{(2(k-1))!}{2^{k-1}(k-1)!}.$$

We use Stirling's approximation. That is,

$$n! \sim n^n \cdot e^{-n} \cdot \sqrt{2\pi n} \quad \text{as } n \to \infty.$$

It follows as $k \to \infty$

$$k^{\frac{3}{2}}\mathbb{P}(\tau = 2k-1) \sim k^{\frac{3}{2}} \frac{1}{2^{2k-1}} \frac{(2(k-1)^{2(k-1)}e^{-2(k-1)}\sqrt{2\pi 2(k-1)}}{k^k e^{-k}\sqrt{2\pi k}\cdot (k-1)^{k-1}e^{-(k-1)}\sqrt{2\pi(k-1)}}$$

$$k^{\frac{3}{2}}\mathbb{P}(\tau = 2k-1) \sim \frac{1}{2\sqrt{\pi}} \cdot k^{\frac{3}{2}} \cdot \sqrt{\frac{k-1}{k(k-1)}} \cdot e \cdot \frac{(k-1)^{k-1}}{k^k}$$

$$k^{\frac{3}{2}}\mathbb{P}(\tau = 2k-1) \sim \frac{1}{2\sqrt{\pi}} \cdot k \cdot \sqrt{k} \cdot \sqrt{\frac{k-1}{k(k-1)}} \cdot e \cdot \frac{(k-1)^{k-1}}{k^k}$$

$$k^{\frac{3}{2}}\mathbb{P}(\tau = 2k-1) \sim \frac{1}{2\sqrt{\pi}} \cdot \underbrace{\sqrt{\frac{k(k-1)}{k(k-1)}}}_{=1} \cdot e \cdot \frac{(k-1)^{k-1}}{k^{k-1}}$$

$$k^{\frac{3}{2}}\mathbb{P}(\tau = 2k-1) \sim \frac{1}{2\sqrt{\pi}} \cdot e \cdot \left(\frac{k-1}{k}\right)^{k-1}$$

$$k^{\frac{3}{2}}\mathbb{P}(\tau = 2k-1) \sim \frac{1}{2\sqrt{\pi}} \cdot e \cdot \underbrace{\left(1 - \frac{1}{k}\right)^{k-1}}_{\to e^{-1}}$$

$$k^{\frac{3}{2}}\mathbb{P}(\tau = 2k-1) \sim \frac{1}{2\sqrt{\pi}}.$$

We conclude

$$\lim_{k\to\infty} k^{\frac{3}{2}}\mathbb{P}(\tau = 2k-1) = \frac{1}{2\sqrt{\pi}}.$$

Or in other words as $k \to \infty$

$$\mathbb{P}(\tau = 2k-1) \sim \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{k^3}}.$$