5th Exercise Sheet

Concentration inequalities I

5.1 (a) Show that the Markov inequality is **sharp**. Namely, for every fixed real numbers $0 < m \leq \lambda$ there exists a random variable X such that $\mathbb{E}(X) = m$ and $\mathbb{P}(X \geq \lambda) = m/\lambda$.

(b) Show that the Markov inequality is **not sharp**. Namely, for every fixed non-negative random variable X with finite expected value $\lim_{\lambda\to\infty} \lambda\,\mathbb{P}(X \geq \lambda)/\mathbb{E}(X) = 0$.

5.2 Let X be a random variable such that $\mathbb{E}(X) = 100$ and $\mathbb{P}(X < 30) = 0$. Give the best estimate for $\mathbb{P}(X > 70)$.

5.3 Let $X_1, \dots, X_9$ be independent random variables with distribution $\mathrm{Uni}(0,1)$. Moreover, let $Y = \sqrt[9]{X_1 \cdots X_9}$. Using Chebysev's inequality, give a lower estimate for the probability

$$\mathbb{P}(e^{-5/3} < Y < e^{-1/3})!$$

5.4 Let $X_1, X_2, \dots$ be random variables with finite variance, and 0 expected value (i.e. for every $i \geq 1$, $\mathbb{E}(X_i) = 0$, $\sigma_i^2 := \mathbb{D}^2(X_i) = \mathbb{E}(X_i^2) < \infty$). Let r_n be a sequence such that $\lim_{n\to\infty} r_n = 0$ and suppose that $\mathbf{Cov}(X_i, X_j) = \mathbb{E}(X_iX_j) \leq r_{|i-j|}$ for every $i, j \geq 1$ (In particular, $\sigma_i^2 \leq r_0$). Let $S_n := X_1 + X_2 + \cdots + X_n$. Show that $\lim_{n\to\infty} \mathbb{P}(|S_n/n| > \delta) = 0$ for every $\delta > 0$.

HW **5.5** Let $X_1, X_2, \dots$ be uncorrelated random variables with finite variance, 0 expected value. (That is, for every $i \geq 1$, $\mathbb{E}(X_i) = 0$, $\sigma_i^2 := \mathbb{D}^2(X_i) = \mathbb{E}(X_i^2) < \infty$, and for every $i \neq j$, $\mathbb{E}(X_iX_j) = 0$). Let $S_n := X_1 + X_2 + \cdots + X_n$. Show that if $\lim_{i\to\infty} \sigma_i^2/i = 0$ then $\lim_{n\to\infty} \mathbb{P}(|S_n/n| > \delta) = 0$ for every $\delta > 0$.

HW* **5.6** *(WLLN for renewal processes)* Let $\tau_1, \tau_2, \dots, \tau_n, \dots$ be i.i.d non-negative random variables. Suppose that $\mathbb{E}(\tau_i) =: m < \infty$. Let $T_n := \sum_{i=1}^n \tau_i$ and let $\nu_t := \max\{n : T_n \leq t\}$. The WLLN states that for every $\delta > 0$, $\lim_{n\to\infty} \mathbb{P}(|T_n/n - m| > \delta) = 0$. Prove that the dual statement is also true, namely, for every $\delta > 0$

$$\lim_{t\to\infty} \mathbb{P}\Big(|\frac{\nu_t}{t} - m^{-1}| > \delta\Big) = 0.$$

Hint: Figure out what $\left|\frac{\nu_t}{t} - m^{-1}\right| > \delta$ states about the random variables T_i.

HW$_2$ **5.7** Let $X_1, X_2, \dots$ be random variables over the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ having the same distribution function $F(x)$. (Nothing else is assumed.) Let $M_n := \max_{1\leq i\leq n} |X_i|$.

(a) Suppose that for some $\alpha > 0$, $\int |x|^\alpha dF(x) < \infty$ (i.e. $\mathbb{E}(|X_i|^\alpha) < \infty$). Prove that for every $\varepsilon > 0$ and $\delta > 0$

$$\lim_{n\to\infty} \mathbb{P}\Big(n^{-(1/\alpha+\varepsilon)}|M_n| > \delta\Big) = 0.$$

(b) Suppose that for some $s > 0$, $\int e^{s|x|} dF(x) < \infty$ (i.e. $\mathbb{E}\Big(e^{s|X_i|}\Big) < \infty$). Prove that for any sequence b_n such that $\lim_{n\to\infty} b_n = \infty$ and for every $\delta > 0$

$$\lim_{n\to\infty} \mathbb{P}\Big((b_n \log n)^{-1}|M_n| > \delta\Big) = 0.$$

Hint: Use the following Markov-like inequality.

$$\mathbb{P}\Big(\max_{1\leq i\leq n} |X_i| > \lambda\Big) = \mathbb{P}(\cup_{i=1}^n \{|X_i| > \lambda\}) \leq \sum_{i=1}^n \mathbb{P}(|X_i| > \lambda) = n\mathbb{P}(|X_i| > \lambda).$$

HW **5.8** (a) Let X be a random variable. We call the function $R(t) = \mathbb{E}\Big(e^{tX}\Big)$ the moment generating function of X. Show that for every $x \in \mathbb{R}$, $\mathbb{P}(X > x) \leq \inf_{t>0} R(t)e^{-tx}$.

(b) Let X be a random variable with distribution $\mathrm{Poi}(\lambda)$. Using the exercise 5.8a, estimate $\mathbb{P}(X > x)$.

5.9 We toss a coin 60 times and denote the number of heads by X. Give an upper bound for the probability

$$\mathbb{P}(|X - 30| \geq 20)$$

by using Chebysev's inequality. A better estimate can be given by using the turbo-Markov inequality:

(a) Let $Y_\beta = e^{\beta X}$, where $0 < \beta$. Show that $\mathbb{E}(Y_\beta) = 2^{-60}(1 + e^\beta)^{60}$.

(b) Give an upper estimate for $\mathbb{P}(X \geq 50)$ by using Markov-inequality for the non-negative random variable Y_β for all $\beta > 0$.

(c) Find the optimal β, that is, find the minimum of the estimate in (b). (This can be done by minimizing the convex function $f(\beta) = \log(1 + e^\beta) - \frac{5}{6}\beta$.)

(d) Combining the previous points, show $\mathbb{P}(|X - 30| \geq 20) \leq 2 \cdot 3^{60} \cdot 5^{-50} < 10^{-6}$.