5th Practice Class
Concentration inequalities I

5.1 (a) Show that the Markov inequality is **sharp**. Namely, for every fixed real numbers $0 < m \le \lambda$ there exists a random variable X such that $\mathbb{E}(X) = m$ and $\mathbb{P}(X \ge \lambda) = m/\lambda$.

(b) Show that the Markov inequality is **not sharp**. Namely, for every fixed non-negative random variable X with finite expected value $\lim_{\lambda\to\infty} \lambda\,\mathbb{P}(X \ge \lambda)/\mathbb{E}(X) = 0$.

Solution

(a) Let $0 < m \le \lambda$ and X such that

$$\mathbb{P}(X = \lambda) = 1 - \mathbb{P}(X = 0) = \frac{m}{\lambda}.$$

Then

$$\mathbb{E}(X) = 0 \cdot \left(1 - \frac{m}{\lambda}\right) + \lambda \cdot \frac{m}{\lambda} = m.$$

and

$$\mathbb{P}(X \ge \lambda) = \mathbb{P}(X = \lambda) = \frac{m}{\lambda} = \frac{\mathbb{E}(X)}{\lambda}.$$

(b) First notice

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(X\mathbb{1}[X \ge \lambda]) + \mathbb{E}(X\mathbb{1}[X < \lambda])\\ &\ge \lambda\mathbb{E}(\mathbb{1}[X \ge \lambda]) + \mathbb{E}(X\mathbb{1}[X < \lambda])\\ &= \lambda\mathbb{P}(X \ge \lambda) + \mathbb{E}(X\mathbb{1}[X < \lambda]).\end{aligned}$$

Therefore,

$$0 \le \lambda\mathbb{P}(X \ge \lambda) \le \mathbb{E}(X) - \mathbb{E}(X\mathbb{1}[X < \lambda]).$$

Moreover, notice that

$$\lim_{\lambda\to\infty} \mathbb{E}(X\mathbb{1}[X < \lambda]) = \mathbb{E}(X).$$

It follows from monotone convergence since $X\mathbb{1}[X < \lambda] \nearrow X$ almost surely as $\lambda \to \infty$. Thus,

$$\lim_{\lambda\to\infty} \lambda\mathbb{P}(X \ge \lambda) = 0.$$

Dividing by $\mathbb{E}(X)$

$$\lim_{\lambda\to\infty} \lambda\mathbb{P}(X \ge \lambda)/\mathbb{E}(X) = 0.$$

5.4 Let $X_1, X_2, \ldots$ be random variables with finite variance, and 0 expected value (i.e. for every $i \ge 1$, $\mathbb{E}(X_i) = 0$, $\sigma_i^2 := \mathbb{D}^2(X_i) = \mathbb{E}(X_i^2) < \infty$). Let r_n be a sequence such that $\lim_{n\to\infty} r_n = 0$ and suppose that $\mathbf{Cov}(X_i, X_j) = \mathbb{E}(X_iX_j) \le r_{|i-j|}$ for every $i, j \ge 1$ (In particular, $\sigma_i^2 \le r_0$). Let $S_n := X_1 + X_2 + \cdots + X_n$. Show that $\lim_{n\to\infty} \mathbb{P}(|S_n/n| > \delta) = 0$ for every $\delta > 0$.

Solution Notice

$$\mathbb{E}(S_n) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \underbrace{\mathbb{E}(X_i)}_{=0} = 0$$

and

$$\mathbb{D}^2(S_n) = \mathbf{Cov}(S_n, S_n) = \mathbf{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=j}^{n} X_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{Cov}(X_i, X_j) \le \sum_{i=1}^{n}\sum_{j=1}^{n} r_{|i-j|}.$$

Now notice $r_{|i-j|}$ only depends on the distance between i and j. The distance is zero in n cases ($i = j$) and the distance is k ($k = 1, \ldots, n-1$) in $2(n-k)$ cases ($i = j + k$ or $j = i + k$). Therefore,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_{|i-j|} = nr_0 + 2\sum_{k=1}^{n-1}(n-k)r_k \le nr_0 + 2n\sum_{k=1}^{n-1} r_k.$$

We can use Chebyshev's inequality for $\delta > 0$

$$\mathbb{P}(|S_n/n| > \delta) = \mathbb{P}(|S_n| > n\delta) = \mathbb{P}(|S_n - \mathbb{E}(S_n)| > n\delta) \leq \frac{1}{\delta^2 n^2}\mathbb{D}^2(S_n).$$

Using the previous bounds on $\mathbb{D}^2(S_n)$ as $n \to \infty$

$$\mathbb{P}(|S_n/n| > \delta) \leq \frac{r_0}{\delta^2} \cdot \frac{1}{n} + \frac{2}{\delta^2} \cdot \frac{1}{n} \sum_{k=1}^{n-1} r_k = \underbrace{\frac{r_0}{\delta^2} \cdot \frac{1}{n}}_{\to 0} + \frac{2}{\delta^2} \cdot \underbrace{\frac{n-1}{n}}_{\to 1} \cdot \underbrace{\frac{1}{n-1} \sum_{k=1}^{n-1} r_k}_{\to 0} \to 0.$$

The last convergence holds since $\frac{1}{n-1}\sum_{k=1}^{n-1} r_k$ is the Césaro average of the sequence (r_n). Therefore, it is convergent and has the same limit

$$\lim_{n\to\infty} \frac{1}{n-1} \sum_{k=1}^{n-1} r_k = \lim_{n\to\infty} r_n = 0.$$

5.9 We toss a coin 60 times and denote the number of heads by X. Give an upper bound for the probability

$$\mathbb{P}(|X - 30| \geq 20)$$

by using Chebysev's inequality. A better estimate can be given by using the turbo-Markov inequality:

(a) Let $Y_\beta = e^{\beta X}$, where $0 < \beta$. Show that $\mathbb{E}(Y_\beta) = 2^{-60}(1 + e^\beta)^{60}$.

(b) Give an upper estimate for $\mathbb{P}(X \geq 50)$ by using Markov-inequality for the non-negative random variable Y_β for all $\beta > 0$.

(c) Find the optimal β, that is, find the minimum of the estimate in (b). (This can be done by minimizing the convex function $f(\beta) = \log(1 + e^\beta) - \frac{5}{6}\beta$.)

(d) Combining the previous points, show $\mathbb{P}(|X - 30| \geq 20) \leq 2 \cdot 3^{60} \cdot 5^{-50} < 10^{-6}$.

Solution We know that $X \sim \text{Bin}(60, 1/2)$. Therefore,

$$\mathbb{E}(X) = 60 \cdot \frac{1}{2} = 30 \qquad \text{and} \qquad \mathbb{D}^2(X) = 60 \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{2}\right) = 15.$$

Chebysev's inequality implies

$$\mathbb{P}(|X - 30| \geq 20) = \mathbb{P}(|X - \mathbb{E}(X)| \geq 20) \leq \frac{\mathbb{D}^2(X)}{20^2} = \frac{15}{400} = \frac{3}{80}.$$

(a)

$$\mathbb{E}(Y_\beta) = \mathbb{E}\left(e^{\beta X}\right) = \sum_{k=0}^{n} e^{\beta k} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{60-k} = 2^{-60} \sum_{k=0}^{n} \binom{n}{k} \left(e^\beta\right)^k = 2^{-60}(1 + e^\beta)^{60}.$$

(b) Applying the "Turbo" Markov inequality with the function $x \mapsto e^{\beta x}$ (its range is in $\mathbb{R}^+$ and it is non-decreasing)

$$\mathbb{P}(X \geq 50) \leq \frac{\mathbb{E}\left(e^{\beta X}\right)}{e^{\beta 50}} = \mathbb{E}(Y_\beta)e^{-50\beta} = 2^{-60}(1 + e^\beta)^{60} e^{-50\beta}.$$

This bound holds for any $\beta > 0$.

(c) The optimal $\beta > 0$ is the one which minimizes the right hand side of the inequality. Moreover, since log is strictly monotone increasing it is enough to minimize

$$\log(2^{-60}(1 + e^\beta)^{60} e^{-50\beta}) = -60\log(2) + 60\log(1 + e^\beta) - 50\beta.$$

Clearly, it attains its minimum at the same β as the function

$$f(\beta) = \log(1 + e^\beta) - \frac{5}{6}\beta.$$

The derivatives are

$$f'(\beta) = \frac{e^\beta}{1+e^\beta} - \frac{5}{6}$$
$$f''(\beta) = \frac{e^\beta}{(1+e^\beta)^2} > 0$$

Notice

$$\begin{aligned} 0 &= f'(\beta) \\ 0 &= e^\beta/(1+e^\beta) - 5/6 \\ e^\beta &= 5 \\ \beta &= \log(5) \end{aligned}$$

Therefore, the best bound using this method is

$$\mathbb{P}(X \geq 50) \leq 2^{-60}(1+e^{\log(5)})^{60} e^{-50\log(5)} = 2^{-60} \cdot 6^{60} \cdot 5^{-50} = 3^{60} \cdot 5^{-50}.$$

(d) Notice that

$$\mathbb{P}(|X-30| \geq 20) = \mathbb{P}(X \leq 10) + \mathbb{P}(X \geq 50).$$

Moreover, since $X \sim \mathrm{Bin}(60, 1/2)$

$$\mathbb{P}(X \leq 10) = \mathbb{P}(X \geq 50).$$

It follows that

$$\mathbb{P}(|X-30| \geq 20) \leq 2 \cdot \mathbb{P}(X \geq 50) = 2 \cdot 3^{60} \cdot 5^{-50} < 10^{-6}.$$