6th Exercise Sheet
Concentration inequalities II

6.1 (a) Let us suppose that we have an unfair die. That is, it has a shape of a general parallelepiped but the sum of the opposite sides are still 7. Let X_i be the outcome of the ith roll (the rolls are independent), and let $S_n = X_1 + \cdots + X_n$. Using Bernstein's, give estimates for

$$\mathbb{P}\left(\left|S_n - n\frac{7}{2}\right| > n\right).$$

(b) Give an estimate by using Hoeffding's inequality!

HW 6.2 Draw 100 strings in a circle with radius 1 independently from each other in such a way that the endpoints of the strings are chosen independently and uniformly on the unit circle. Denote the total length of the strings by X. Using Bernstein's inequality, give a lower estimate for the probability

$$\mathbb{P}\left(\left|X - \frac{400}{\pi}\right| < 10\frac{\sqrt{8\pi^2 - 64}}{\pi}\right).$$

6.3 Let X be a random variable. The function $\widehat{I}(\lambda) = \log(\mathbb{E}\left(e^{\lambda X}\right))$ is called logarithmic moment generating function.

(a) Find the logarithmic moment generating function of $aX + b$.

(b) Let X_1, X_2 be independent random variables, find the LMGF of $X_1 + X_2$.

(c) Let $X_1, X_2, \ldots$ be i.i.d. random variables and let ν be $\mathbb{N}$-valued random variable independent of X_is. Let $Y = \sum_{n=1}^{\nu} X_i$. Express the logarithmic moment generating function of Y by the LMGF of ν and X_1.

6.4 Let us define the Legendre transform $\hat{f}$ of a function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that

$$\hat{f}(x) = \sup_{\lambda \in \mathbb{R}}\{x\lambda - f(\lambda)\}.$$

(a) Show that if f is affine linear then $\hat{\hat{f}} = f$.

(b) Show that if f is C^2 and strictly convex then the derivative of the Legendre transform of f equals to the inverse of the derivative of f and moreover, $\hat{\hat{f}} = f$.

6.5 Let us define the *rate function* of the random variable X by the Legendre transform of the $I(x) = \sup_{\lambda \in \mathbb{R}}\{\lambda x - \widehat{I}(\lambda)\}$, where $\widehat{I}(\lambda) = \log(\mathbb{E}\left(e^{\lambda X}\right))$ is the logarthmic moment generating function of X. Suppose that $\widehat{I}(\lambda)$ exists for every $\lambda \in \mathbb{R}$

(a) Using the strict convexity of $\widehat{I}$ show that $I(x) = x \cdot (\widehat{I}')^{-1}(x) - \widehat{I}((\widehat{I}')^{-1}(x))$.

(b) Show that I is C^∞, $I(\mathbb{E}(X)) = 0$ and $I(x) > 0$ for every $x \neq \mathbb{E}(X)$.

(c) Find the logarithmic moment generating function of random variables with distributions Bernoulli, Binomial, Poisson, Exponential, Geometric, Normal (with the range)!

(d) Find the rate function of the random variables with distributions Bernoulli, Binomial, Poisson, Exponential, Geometric and Normal (with the range)! (Hint: Use the learnt formula that $I(x) = x \cdot (\widehat{I}')^{-1}(x) - \widehat{I}((\widehat{I}')^{-1}(x))$.)

6.6 *(Large deviation for renewal processes)* Let $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ be i.i.d non-negative random variables. Suppose that $\mathbb{E}(\tau_i) =: m < \infty$. Let $T_n := \sum_{i=1}^{n} \tau_i$ and let $\nu_t := \max\{n : T_n \leq t\}$. Denote by $I(x)$ the rate function of τ_1 and denote by $\widehat{I}(\lambda)$ the logarithmic moment generating function of τ_1. For every $y_1 \leq \frac{1}{m} \leq y_2$, using the rate function I find bounds for

$$\mathbb{P}\left(\frac{\nu_t}{t} \leq y_1\right) \text{ for } t \gg 0 \text{ and } \mathbb{P}\left(\frac{\nu_t}{t} \geq y_2\right) \text{ for } t \ll 0.$$

6.7 (a) Using Stirling's Formula, show Cramér's Theorem for the sum of independent indicators, i.e. show if $S_n \sim \mathrm{BIN}(n, p)$ then

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}(S_n = \lceil nx \rceil) = I(x) = x \log\left(\frac{x}{p}\right) + (1-x)\log\left(\frac{1-x}{1-p}\right).$$

(b) By using (a), show $\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}\left(\frac{S_n}{n} \geq x\right) = I(x)$ for $x \geq p$.

(c) Show that $I(x)$ is the Legendre transform of $\hat{I}(\lambda) = \log(1 - p + pe^\lambda)$.

HW$_2$ **6.8** (a) Let X be a random variable with standard normal distribution $N(0,1)$. Prove that for $x > 0$

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}} \le \mathbb{P}(X \ge x) \le \frac{1}{x}\frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

(Hint: Compare the derivatives.)

(b) Using exercise **6.8**a, show the special case of Cramér's Theorem for i.i.d. $N(0,1)$ random variables. That is, let $X_1, X_2, \cdots$ be i.i.d random variables such that X_i has distribution $N(0,1)$. Show that

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathbb{P}\left(\frac{X_1 + \cdots + X_n}{n} \in [a,b]\right) = \inf_{x\in[a,b]} \frac{x^2}{2}.$$

6.9 (*The Bernstein inequality is asymptotically sharp*) Let $X_1, X_2, \ldots$ be i.i.d random variables with $\mathbb{P}(X_i = \pm 1) = \frac{1}{2}$, and let $Y_n = \frac{X_1+\cdots+X_n}{\sqrt{n}}$.

(a) Give an upper bound for $\lim_{n\to\infty} \mathbb{P}(|Y_n| \ge \lambda)$ by using Bernstein's inequality.

(b) Show that $\mathbb{P}(|Y_n| \ge \lambda) \le 2\exp(-nI(\frac{\lambda}{\sqrt{n}}))$, where

$$I(x) = -\frac{1+x}{2}\log(1+x) - \frac{1-x}{2}\log(1-x)$$

is the rate function of the Bernoulli distribution of X_i.

(c) Using **6.9**b, give an upper bound for $\lim_{n\to\infty} \mathbb{P}(|Y_n| \ge \lambda)$.

(d) Show by using De Moivre's Central Limit Theorem and refex:normal that for every $\varepsilon > 0$ there exists λ_0 such that for every $\lambda \ge \lambda_0$

$$\lim_{n\to\infty} \mathbb{P}(|Y_n| \ge \lambda) \ge 2e^{-(1+\varepsilon)\frac{\lambda^2}{2}}.$$