7th Exercise Sheet

Types of convergences I

7.1 Let $X_1, X_2, \ldots, X_n, \ldots, Y_1, Y_2, \ldots, Y_n, \ldots, X$ and Y be random variables on the same prob. space $(\Omega, \mathcal{A}, \mathbb{P})$ and suppose that $X_n \xrightarrow{\mathbb{P}} X$ and $Y_n \xrightarrow{\mathbb{P}} Y$. Show

(a) $aX_n \xrightarrow{\mathbb{P}} aX$ $(a \in \mathbb{R})$,

(b) $X_n \pm Y_n \xrightarrow{\mathbb{P}} X \pm Y$,

(c) $X_n Y_n \xrightarrow{\mathbb{P}} XY$,

(d) If $\mathbb{P}(Y_n \neq 0) = 1$ then $X_n/Y_n \xrightarrow{\mathbb{P}} X/Y$.

(e) If $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f(X_n) \xrightarrow{\mathbb{P}} f(X)$.

7.2 Let $X_1, X_2, \ldots$ and Y be random variables on the same prob. space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $F_1(x)$, $F_2(x)$, ... and $G(x)$ be their distribution functions respectively. Show that if $X_n \xrightarrow{\mathbb{P}} Y$ then $\lim_{n\to\infty} F_n(x) = G(x)$ at every continuity point x of G.

7.3 Let H be a subset of L^1 random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that the set H is uniformly integrable if

$$\lim_{M\to\infty} \sup_{X\in H} \mathbb{E}(|X| \mathbb{1}_{\{|X|>M\}}) = 0.$$

(a) Show that every finite subset of $L^1(\Omega, \mathbb{P})$ is uniformly integrable.

(b) If $Y \in L^1$ then the set $\{X \in L^1(\Omega) : |X| < |Y|\}$ is uniformly integrable.

(c) If H is uniformly integrable then $\{X \in L^1(\Omega) : \exists Y \in H \ |X| \leq |Y|\}$ is uniformly integrable.

HW* **7.4** Show that a set H of random variables is uniformly integrable if and only if the followings hold:

(1) $\sup_{X\in H} \mathbb{E}(|X|) < \infty$

(2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathbb{E}(|X| \mathbb{1}_A) \leq \varepsilon$ for every $\mathbb{P}(A) \leq \delta$ and every $X \in H$.

7.5 Show that if $X_n \xrightarrow{L^1} X$, then $\mathbb{E}(X_n) \to \mathbb{E}(X)$. Furthermore, if $\mathbb{E}(X_n) \to \mathbb{E}(X)$ and $\mathbb{P}(X_n \leq X) = 1$ for every n, then $X_n \xrightarrow{L^1} X$.

7.6 Let $X_1, X_2, X_3, \ldots$ be random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $Y_n := \sup_{m\geq n} |X_m|$. Show that the following statements are equivalent:

(1) $X_n \xrightarrow{\text{a.s.}} 0$, as $n \to \infty$.

(2) $Y_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$.