# 7th Practice class

## Types of convergences I

**7.1** Let $X_1, X_2, \dots, X_n, \dots,$ $Y_1, Y_2, \dots, Y_n, \dots,$ $X$ and $Y$ be random variables on the same prob. space $(\Omega, \mathcal{A}, \mathbb{P})$ and suppose that $X_n \xrightarrow{\mathbb{P}} X$ and $Y_n \xrightarrow{\mathbb{P}} Y$. Show

(a) $aX_n \xrightarrow{\mathbb{P}} aX$ $(a \in \mathbb{R})$,

(b) $X_n \pm Y_n \xrightarrow{\mathbb{P}} X \pm Y$,

(c) $X_n Y_n \xrightarrow{\mathbb{P}} XY$,

(d) If $\mathbb{P}(Y_n \neq 0) = 1$ then $X_n / Y_n \xrightarrow{\mathbb{P}} X/Y$.

(e) If $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f(X_n) \xrightarrow{\mathbb{P}} f(X)$.

**Solution**

(a) Let $\delta > 0$. For $a = 0$, it is clearly true. Suppose $a \neq 0$.

$$\mathbb{P}(|aX_n - aX| \geq \delta) = \mathbb{P}(|a|\,|X_n - X| \geq \delta) = \mathbb{P}\Big(|X_n - X| \geq \frac{\delta}{|a|}\Big) \to 0.$$

(b) Let $\delta > 0$.

$$\mathbb{P}(|(X_n + Y_n) - (X + Y)| \geq \delta)$$
$$= \mathbb{P}(|(X_n - X) + (Y_n - Y)| \geq \delta) \overset{(*)}{\leq} \mathbb{P}\Big(\Big\{|X_n - X| \geq \frac{\delta}{2}\Big\} \bigcup \Big\{|Y_n - Y| \geq \frac{\delta}{2}\Big\}\Big)$$

Since by the triangle inequality

$$\Big\{|X_n - X| < \frac{\delta}{2}\Big\} \bigcup \Big\{|Y_n - Y| < \frac{\delta}{2}\Big\} \subseteq \{|(X_n - X) + (Y_n - Y)| < \delta\},$$

for the complements we have

$$\{|(X_n - X) + (Y_n - Y)| \geq \delta\} \subseteq \Big\{|X_n - X| \geq \frac{\delta}{2}\Big\} \bigcup \Big\{|Y_n - Y| \geq \frac{\delta}{2}\Big\}.$$

Therefore, the inequality $(*)$ holds. Using the union bound,

$$\mathbb{P}\Big(\Big\{|X_n - X| \geq \frac{\delta}{2}\Big\} \bigcup \Big\{|Y_n - Y| \geq \frac{\delta}{2}\Big\}\Big) \leq \mathbb{P}\Big(|X_n - X| \geq \frac{\delta}{2}\Big) + \mathbb{P}\Big(|Y_n - Y| \geq \frac{\delta}{2}\Big).$$

Hence,

$$\mathbb{P}(|(X_n + Y_n) - (X + Y)| \geq \delta) \leq \underbrace{\mathbb{P}\Big(|X_n - X| \geq \frac{\delta}{2}\Big)}_{\to 0} + \underbrace{\mathbb{P}\Big(|Y_n - Y| \geq \frac{\delta}{2}\Big)}_{\to 0} \to 0.$$

(e) Let $\varepsilon, \varrho > 0$. There exists $M > 0$ such that

$$\mathbb{P}(|X| > M) < \varepsilon.$$

Since $[-(M+1), M+1]$ is compact and $f$ is continuous, there exits a $0 < \delta < 1$ such that for any $x, y \in [-(M+1), M+1]$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varrho.$$

Then

$$\mathbb{P}(|f(X_n) - f(X)| \geq \varrho)$$
$$\leq \mathbb{P}\Big(\{|X_n - X| \geq \delta\} \bigcup \{|X| > M\}\Big) \leq \underbrace{\mathbb{P}(|X_n - X| \geq \delta)}_{\to 0} + \underbrace{\mathbb{P}(|X| > M)}_{<\varepsilon}.$$

Thus, for any $\varepsilon, \varrho > 0$

$$0 \leq \limsup_{n\to\infty} \mathbb{P}(|f(X_n) - f(X)| \geq \varrho) \leq \varepsilon.$$

It follows for any $\varrho > 0$

$$\lim_{n\to\infty} \mathbb{P}(|f(X_n) - f(X)| \geq \varrho) = 0.$$

**7.2** Let $X_1, X_2, \dots$ and $Y$ be random variables on the same prob. space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $F_1(x)$, $F_2(x)$, ... and $G(x)$ be their distribution functions respectively. Show that if $X_n \xrightarrow{\mathbb{P}} Y$ then $\lim_{n\to\infty} F_n(x) = G(x)$ at every continuity point $x$ of $G$.

**Solution** Let $\varepsilon > 0$ and $x \in \mathbb{R}$ a point of continuity of $G$.

(i) First notice

$$\begin{aligned} F_n(x) \leq F_n(x+\varepsilon) = \mathbb{P}(X_n < x+\varepsilon) &\overset{(*)}{\leq} \mathbb{P}\Big(\{|X_n - X| > \varepsilon\} \bigcup \{X < x + 2\varepsilon\}\Big) \\ &\overset{(**)}{\leq} \mathbb{P}(|X_n - X| > \varepsilon) + \mathbb{P}(X < x + 2\varepsilon) = \underbrace{\mathbb{P}(|X_n - X| > \varepsilon)}_{\to 0} + G(x+2\varepsilon). \end{aligned}$$

The inequality $(*)$ holds since

$$\underbrace{\{-\varepsilon \leq X_n - X \leq \varepsilon\}}_{\{|X_n - X| \leq \varepsilon\}} \bigcap \{X \geq x + 2\varepsilon\} \subseteq \{X_n \geq x + \varepsilon\}.$$

Thus, for the complements

$$\{X_n < x + \varepsilon\} \subseteq \{|X_n - X| > \varepsilon\} \bigcup \{X < x + 2\varepsilon\}.$$

We then applied the union bound in $(**)$. Hence,

$$\limsup_{n\to\infty} F_n(x) \leq G(x + 2\varepsilon).$$

(ii) Using similar arguments as in (i)

$$\begin{aligned} F_n(x) \geq F_n(x-\varepsilon) = \mathbb{P}(X_n < x-\varepsilon) &\leq \mathbb{P}\Big(\{|X_n - X| > \varepsilon\} \bigcup \{X < x - 2\varepsilon\}\Big) \\ &\leq \mathbb{P}(|X_n - X| > \varepsilon) + \mathbb{P}(X < x - 2\varepsilon) = \underbrace{\mathbb{P}(|X_n - X| > \varepsilon)}_{\to 0} + G(x-2\varepsilon). \end{aligned}$$

Hence,

$$\liminf_{n\to\infty} F_n(x) \geq G(x - 2\varepsilon).$$

Sice $G$ is continuous at $x$, as $\varepsilon \to 0$

$$\underbrace{G(x - 2\varepsilon)}_{\to G(x)} \leq \liminf_{n\to\infty} F_n(x) \leq \limsup_{n\to\infty} F_n(x) \leq \underbrace{G(x + 2\varepsilon)}_{\to G(x)}.$$

Thus,

$$\lim_{n\to\infty} F_n(x) = G(x).$$

**7.5** Show that if $X_n \xrightarrow{L^1} X$, then $\mathbb{E}(X_n) \to \mathbb{E}(X)$. Furthermore, if $\mathbb{E}(X_n) \to \mathbb{E}(X)$ and $\mathbb{P}(X_n \leq X) = 1$ for every $n$, then $X_n \xrightarrow{L^1} X$.

**Solution** Suppose $X_n \xrightarrow{L^1} X$. Since $x \mapsto |x|$ is a convex function, applying Jensen's inequality at $(*)$ we get

$$|\mathbb{E}(X_n) - \mathbb{E}(X)| = |\mathbb{E}(X_n - X)| \overset{(*)}{\leq} \mathbb{E}(|X_n - X|) \to 0.$$

Suppose $X_n \leq X$ almost surely and $\mathbb{E}(X_n) \to \mathbb{E}(X)$. Then $X - X_n \geq 0$ almost surely. Hence,

$$\mathbb{E}(|X_n - X|) = \mathbb{E}(|X - X_n|) = \mathbb{E}(X - X_n) = \mathbb{E}(X) - \mathbb{E}(X_n) \to 0.$$

**7.6** Let $X_1, X_2, X_3, \dots$ be random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $Y_n := \sup_{m \geq n} |X_m|$. Show that the following statements are equivalent:

(1) $X_n \xrightarrow{\text{a.s.}} 0$, as $n \to \infty$.

(2) $Y_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$.

**Solution** Since $\bigcap_{m \geq n}^{\infty} \Big\{ |X_n| \leq \frac{1}{N} \Big\} = \Big\{ Y_n \leq \frac{1}{N} \Big\}$ for any $N \geq 1$,

$$(1) \Leftrightarrow \mathbb{P}\Big( \lim_{n \to \infty} X_n = 0 \Big) \Leftrightarrow \mathbb{P}\left( \bigcap_{N=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{m \geq n}^{\infty} \left\{ |X_n| \leq \frac{1}{N} \right\} \right) \Leftrightarrow \mathbb{P}\left( \bigcap_{N=1}^{\infty} \bigcup_{m=1}^{\infty} \left\{ Y_n \leq \frac{1}{N} \right\} \right).$$

Since the intersection has probability one if and only if all events have probability one,

$$\mathbb{P}\left( \bigcap_{N=1}^{\infty} \bigcup_{m=1}^{\infty} \left\{ Y_n \leq \frac{1}{N} \right\} \right) \Leftrightarrow \forall N \geq 1 : \; 1 = \mathbb{P}\left( \bigcup_{n=1}^{\infty} \left\{ Y_n \leq \frac{1}{N} \right\} \right).$$

Since the events are increasing in $n$: $\Big\{ Y_n \leq \frac{1}{N} \Big\} \subseteq \Big\{ Y_{n+1} \leq \frac{1}{N} \Big\}$, by the continuity of probability

$$\forall N \geq 1 : \; 1 = \mathbb{P}\left( \bigcup_{n=1}^{\infty} \left\{ Y_n \leq \frac{1}{N} \right\} \right) \Leftrightarrow \forall N \geq 1 : \; 1 = \lim_{n \to \infty} \mathbb{P}\left( \left\{ Y_n \leq \frac{1}{N} \right\} \right) \Leftrightarrow Y_n \xrightarrow{\mathbb{P}} 0 \Leftrightarrow (2).$$