8th Exercise Sheet
Types of convergences II

HW 8.1 Show that the convergence in probability is metrisable with respect to

$$\rho(X,Y) := \inf\{\varepsilon > 0 : \mathbb{P}(|X-Y| > \varepsilon) < \varepsilon\}.$$

That is, show that ρ is a metric and $X_n \xrightarrow{\mathbb{P}} Y$ if and only if $\rho(X_n, Y) \to 0$. (Remark: This metric is complete.)

8.2 Find the value of the next limit.

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1\, dx_2 \cdots dx_n$$

HW 8.3 Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that

(a) $$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) dx_1\, dx_2 \cdots dx_n = f\left(\frac{1}{2}\right),$$

(b) $$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \cdots x_n)^{1/n}\right) dx_1\, dx_2 \cdots dx_n = f\left(\frac{1}{e}\right).$$

8.4 Let $f\colon [0,1] \mapsto \mathbb{R}$ be bounded, three times continuously differentiable function. Find the value of the next limit.

$$\lim_{n\to\infty} n \int_0^1 \cdots \int_0^1 \left(f\left(\frac{x_1 + \cdots + x_n}{n}\right) - f\left(\frac{1}{2}\right)\right) dx_1 \cdots dx_n$$

(Hint: Approximate with Taylor polynomials.)

8.5 Let $X_1, X_2, \ldots, X_n, \ldots$ and X be random variables over the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that the following statements are equivalent.

(i) $X_n \xrightarrow{\mathbb{P}} X$ as $n \to \infty$.

(ii) For every subsequence $k \mapsto n_k$ there exists a sub-subsequence $j \mapsto n_{k_j}$ such that $X_{n_{k_j}} \xrightarrow{\text{a.s.}} X$ as $j \to \infty$.

8.6 *Dominated convergence theorem* Let X_n be a sequence of random variables such that $X_n \xrightarrow{\text{a.s.}} X$ and there exists a random variable Y such that $\mathbb{E}(|Y|) < \infty$ and $\mathbb{P}(|X_n| \le |Y|) = 1$ for every n. Then $\mathbb{E}(X_n) \to \mathbb{E}(X)$ as $n \to \infty$.

HW 8.7 *Scheffé's Theorem* Let X_n be a sequence of non-negative random variables and let X be a random variable such that $X_n \xrightarrow{\text{a.s.}} X$ and $\mathbb{E}(X_n) \to \mathbb{E}(X) < \infty$ as $n \to \infty$. Prove that $X_n \xrightarrow{L^1} X$. (Hint: Use that $|Y| = Y + 2\max\{0, -Y\}$ and the dominated convergence theorem)