8th Practice class
Types of convergences II

8.2 Find the value of the next limit.
$$\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 \frac{x_1^2+x_2^2+\cdots+x_n^2}{x_1+x_2+\cdots+x_n}dx_1\,dx_2\,\cdots\,dx_n$$

Solution Let us denote
$$I_n=\int_0^1\int_0^1\cdots\int_0^1 \frac{x_1^2+x_2^2+\cdots+x_n^2}{x_1+x_2+\cdots+x_n}dx_1\,dx_2\,\cdots\,dx_n.$$
Notice, if $X_1, X_2, \dots$ are i.i.d. $\mathrm{Uni}(0,1)$, then denoting
$$Y_n=\frac{X_1^2+X_2^2+\cdots+X_n^2}{X_1+X_2+\cdots+X_n}$$
we have
$$I_n=\mathbb{E}(Y_n).$$
Moreover, by the Weak law of large numbers
$$Y_n=\underbrace{\frac{X_1^2+X_2^2+\cdots+X_n^2}{n}}_{\xrightarrow{\mathbb{P}}\mathbb{E}(X_1^2)}\cdot\underbrace{\frac{n}{X_1+X_2+\cdots+X_n}}_{\xrightarrow{\mathbb{P}}\mathbb{E}(X_1)^{-1}}\xrightarrow{\mathbb{P}}\frac{\mathbb{E}(X_1^2)}{\mathbb{E}(X_1)}=\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}.$$
Notice that the sequence of variables (Y_n) is uniformly integrable. Since $X_1, X_2, \dots, X_n \in [0,1]$,
$$0\le Y_n=\frac{X_1^2+X_2^2+\cdots+X_n^2}{X_1+X_2+\cdots+X_n}\le\frac{X_1+X_2+\cdots+X_n}{X_1+X_2+\cdots+X_n}=1.$$
Therefore, for $M>1$ we have
$$\sup_{n\ge1}\mathbb{E}\left(|Y_n|\cdot\underbrace{\mathbb{1}\{|Y_n|>M\}}_{=0}\right)=0.$$
Hence,
$$\lim_{M\to\infty}\sup_{n\ge1}\mathbb{E}(|Y_n|\cdot\mathbb{1}\{|Y_n|>M\})=0.$$
Thus,
$$Y_n\xrightarrow{\mathbb{P}}\frac{2}{3}\quad\text{and}\quad(Y_n)\text{ is UI}\;\Leftrightarrow\;Y_n\xrightarrow{L^1}\frac{2}{3}.$$
In particular,
$$I_n=\mathbb{E}(Y_n)\to\mathbb{E}\left(\frac{2}{3}\right)=\frac{2}{3}.$$

8.6 *Dominated convergence theorem* Let X_n be a sequence of random variables such that $X_n\xrightarrow{\text{a.s.}}X$ and there exists a random variable Y such that $\mathbb{E}(|Y|)<\infty$ and $\mathbb{P}(|X_n|\le|Y|)=1$ for every n. Then $\mathbb{E}(X_n)\to\mathbb{E}(X)$ as $n\to\infty$.

Solution

(i) Since by assumption $X_n+|Y|\ge0$ almost surely, applying Fatou's lemma and using that $X_n\xrightarrow{\text{a.s.}}X$
$$\begin{aligned}\mathbb{E}\Big(\liminf_{n\to\infty}X_n+|Y|\Big)&\le\liminf_{n\to\infty}\mathbb{E}(X_n+|Y|)\\ \mathbb{E}(X+|Y|)&\le\mathbb{E}(|Y|)+\liminf_{n\to\infty}\mathbb{E}(X_n)\\ \mathbb{E}(X)+\mathbb{E}(|Y|)&\le\mathbb{E}(|Y|)+\liminf_{n\to\infty}\mathbb{E}(X_n).\end{aligned}$$
Since $\mathbb{E}(|Y|)<\infty$, it follows that
$$\mathbb{E}(X)\le\liminf_{n\to\infty}\mathbb{E}(X_n).$$

(ii) Since by assumption $|Y|-X_n \geq 0$ almost surely, applying Fatou's lemma and using that $X_n \xrightarrow{\text{a.s.}} X$

$$\begin{aligned}\mathbb{E}\Big(\liminf_{n\to\infty} |Y| - X_n\Big) &\leq \liminf_{n\to\infty} \mathbb{E}(|Y| - X_n)\\ \mathbb{E}(|Y| - X) &\leq \mathbb{E}(|Y|) + \liminf_{n\to\infty} -\mathbb{E}(X_n)\\ \mathbb{E}(|Y|) - \mathbb{E}(X) &\leq \mathbb{E}(|Y|) - \limsup_{n\to\infty} \mathbb{E}(X_n).\end{aligned}$$

Since $\mathbb{E}(|Y|) < \infty$, it follows that

$$\begin{gathered}-\mathbb{E}(X) \leq -\limsup_{n\to\infty} \mathbb{E}(X_n)\\ \mathbb{E}(X) \geq \limsup_{n\to\infty} \mathbb{E}(X_n).\end{gathered}$$

From (i) and (ii), we conclude

$$\mathbb{E}(X) \leq \liminf_{n\to\infty} \mathbb{E}(X_n) \leq \limsup_{n\to\infty} \mathbb{E}(X_n) \leq \mathbb{E}(X).$$

Hence,

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X).$$