9th Practice Class

Borel Cantelli lemmas and Kolmogorov's Strong Law of Large Numbers

9.1 Let $X_1, X_2, \dots$ be independent random variables such that

$$\mathbb{P}\Big(X_n = n^2 - 1\Big) = n^{-2}, \quad \mathbb{P}(X_n = -1) = 1 - n^{-2}.$$

Prove that for every $n \in \mathbb{N}$, $\mathbb{E}(X_n) = 0$ but

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \dots + X_n}{n} = -1 \quad \text{almost surely.}$$

Solution The expectations are

$$\mathbb{E}(X_n) = (n^2-1)\cdot\frac{1}{n^2} - 1\cdot\left(1 - \frac{1}{n^2}\right) = 0.$$

Notice that

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n = n^2 - 1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty.$$

Hence, by the first Borel-Cantelli lemma

$$\mathbb{P}(X_n = n^2 - 1 \text{ infinitely often}) = 0.$$

It follows that

$$\mathbb{P}(\lim_{n\to\infty} X_n = -1) = 1.$$

Therefore, by Césaro averaging

$$1 = \mathbb{P}(\lim_{n\to\infty} X_n = -1) \le \mathbb{P}\left(\lim_{n\to\infty} \frac{X_1 + \dots + X_n}{n} = -1\right)$$

which exactly means

$$\lim_{n\to\infty} \frac{X_1 + \dots + X_n}{n} = -1 \qquad \text{almost surely.}$$

9.2 Let X_n be i.i.d random variables with $X_n \sim \text{Geo}(p)$. That is $\mathbb{P}(X_n = k) = p(1-p)^k$ for $k \ge 0$. Show that $\limsup_{n\to\infty} \frac{X_n}{\log n} = |\log(1-p)|^{-1}$ almost surely.

Solution First notice that the event $\limsup_{n\to\infty} \frac{X_n}{|\log(n)|} = |\log(1-p)|^{-1}$ exactly means that for any $N \ge 1$

$$\frac{X_n}{\log(n)} \ge |\log(1-p)|^{-1} + N^{-1} \qquad \text{for only finitely many } n \tag{1}$$

and

$$\frac{X_n}{\log(n)} \ge |\log(1-p)|^{-1} - N^{-1} \qquad \text{for infinitely many } n. \tag{2}$$

(1) We first show that (1) happens almost surely. Notice

$$\begin{aligned}
\mathbb{P}\left(\frac{X_n}{\log(n)} \ge |\log(1-p)|^{-1} + N^{-1}\right) &= \mathbb{P}\left(X_n \ge \frac{\log(n)}{|\log(1-p)|} + \frac{\log(n)}{N}\right) \\
&= (1-p)^{\left\lceil \frac{\log(n)}{|\log(1-p)|} + \frac{\log(n)}{N} \right\rceil} \\
&\le (1-p)^{\frac{\log(n)}{|\log(1-p)|} + \frac{\log(n)}{N}} \\
&= \exp\left(\log(1-p)\left(\frac{\log(n)}{|\log(1-p)|} + \frac{\log(n)}{N}\right)\right) \\
&= \exp\left(\log(n)\left(-1 + \log(1-p)N^{-1}\right)\right) \\
&= n^{-1+\log(1-p)N^{-1}} \\
&= \frac{1}{n^{1-\log(1-p)N^{-1}}}.
\end{aligned}$$

Since $\log(1-p) < 0$, it follows that $1 - \log(1-p)N^{-1} > 1$. Hence,

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\frac{X_n}{\log(n)} \geq |\log(1-p)|^{-1} + N^{-1}\Big) \leq \sum_{n=1}^{\infty} \frac{1}{n^{1-\log(1-p)N^{-1}}} < \infty.$$

Therefore, by the first Borel-Cantelli lemma

$$\mathbb{P}\Big(\frac{X_n}{\log(n)} \geq |\log(1-p)|^{-1} + N^{-1} \text{ for infinitely many } n\Big) = 0.$$

As a consequence,

$$\mathbb{P}\Big(\frac{X_n}{\log(n)} \geq |\log(1-p)|^{-1} + N^{-1} \text{ for only finitely many } n\Big) = 1.$$

In particular, we showed that (by taking the intersection over $N \geq 1$)

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{X_n}{\log(n)} \leq |\log(1-p)|^{-1}\Big) = 1.$$

(2) We next show that (2) happens almost surely. Notice

$$\begin{aligned}
\mathbb{P}\Big(\frac{X_n}{\log(n)} \geq |\log(1-p)|^{-1} - N^{-1}\Big) &= \mathbb{P}\Big(X_n \geq \frac{\log(n)}{|\log(1-p)|} - \frac{\log(n)}{N}\Big) \\
&= (1-p)^{\left\lceil \frac{\log(n)}{|\log(1-p)|} - \frac{\log(n)}{N} \right\rceil} \\
&\geq (1-p)^{\frac{\log(n)}{|\log(1-p)|} - \frac{\log(n)}{N} + 1} \\
&= (1-p)\exp\Big(\log(1-p)\Big(\frac{\log(n)}{|\log(1-p)|} - \frac{\log(n)}{N}\Big)\Big) \\
&= (1-p)\exp\Big(\log(n)\Big(-1 - \log(1-p)N^{-1}\Big)\Big) \\
&= (1-p)n^{-1-\log(1-p)N^{-1}} \\
&= (1-p)\frac{1}{n^{1+\log(1-p)N^{-1}}}.
\end{aligned}$$

Since $\log(1-p) < 0$, it follows that $1 + \log(1-p)N^{-1} < 1$. Hence,

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\frac{X_n}{\log(n)} \geq |\log(1-p)|^{-1} + N^{-1}\Big) \geq (1-p)\sum_{n=1}^{\infty} \frac{1}{n^{1+\log(1-p)N^{-1}}} = \infty.$$

Since the variables are independent, by the second Borel-Cantelli lemma it follows that

$$\mathbb{P}\Big(\frac{X_n}{\log(n)} \geq |\log(1-p)|^{-1} - N^{-1} \text{ for infinitely many } n\Big) = 1.$$

In particular, we showed that (by taking the intersection over $N \geq 1$)

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{X_n}{\log(n)} \geq |\log(1-p)|^{-1}\Big) = 1.$$

We conclude

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{X_n}{\log(n)} = \frac{1}{|\log(1-p)|}\Big) = 1.$$

9.4 Let $X_1, X_2, \ldots$ be independent random variables such that $\mathbb{P}(X_n = 1) = p_n$ and $\mathbb{P}(X_n = 0) = 1 - p_n$. Which properties does $p_n, n = 1, 2, \ldots$ have if

(a) $X_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$

(b) $X_n \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$.

Solution

(a) Notice that

$$X_n \xrightarrow{\mathbb{P}} 0 \;\Leftrightarrow\; \forall \delta > 0 \;\mathbb{P}(|X_n| \geq 0) \to 0 \;\Leftrightarrow\; \forall 0 < \delta < 1 \;\mathbb{P}(X_n \geq \delta) = p_n \to 0.$$

Therefore,

$$X_n \xrightarrow{\mathbb{P}} 0 \quad \text{if and only if} \quad \lim_{n\to\infty} p_n = 0.$$

(b) If we have

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n = 1) = \sum_{n=1}^{\infty} p_n < \infty,$$

then by the first Borel-Cantelli lemma it follows that

$$\mathbb{P}(X_n = 1 \text{ infinitely often}) = 0.$$

As a consequence

$$\mathbb{P}\Big(\lim_{n\to\infty} X_n = 0\Big) = 1.$$

However, if we have

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n = 1) = \sum_{n=1}^{\infty} p_n = \infty,$$

then since the variables are independent, by the second Borel-Cantelli lemma it follows that

$$\mathbb{P}(X_n = 1 \text{ infinitely often}) = 1.$$

As a consequence

$$\mathbb{P}\Big(\lim_{n\to\infty} X_n \neq 0\Big) = 1.$$

We conclude

$$X_n \xrightarrow{\text{a.s.}} 0 \quad \text{if and only if} \quad \sum_{n=1}^{\infty} p_n < \infty.$$

9.10 Let $f : [0,1] \to \mathbb{R}$ be a continuous integrable (i.e. $\int_0^1 |f(x)|\, dx < \infty$) function. Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution Uni$(0,1)$. Show that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{f(X_1) + \cdots + f(X_n)}{n} = \int_0^1 f(x)\, dx\right) = 1.$$

Solution Since f is continuous, the random variables $f(X_1), \dots f(X_n)$ are well-defined and they are independent and identically distributed. Moreover, by assumption

$$\mathbb{E}(|f(X_k)|) = \int_0^1 |f(x)|\, \mathrm{d}x < \infty.$$

Hence, by Kolmogorov's Strong Law of Large Numbers

$$\frac{f(X_1) + \cdots + f(X_n)}{n} \xrightarrow{\text{a.s.}} \mathbb{E}(f(X_1)) = \int_0^1 f(x)\, \mathrm{d}x.$$

9.12 (Longest sequence of heads I.) Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution $\mathbb{P}(X_k = 1) = p$, $\mathbb{P}(X_k = 0) = q$, where $p + q = 1$. Let us fix a parameter $\lambda > 1$ and denote $A_k^{(\lambda)}$ for $k = 0, 1, 2, \dots$ the following event

$$A_k^{(\lambda)} := \Big\{\exists r \in \Big[[\lambda^k], [\lambda^{k+1}] - k\Big] \cap \mathbb{N} : X_r = X_{r+1} = \cdots = X_{r+k-1} = 1\Big\}.$$

In particular, the event $A_k^{(\lambda)}$ means that between $[\lambda^k]$ and $[\lambda^{k+1}] - 1$ there exists somewhere a sequence containing only 1 and with length k Show that

(a) If $\lambda < p^{-1}$ then $A_k^{(\lambda)}$ happen for at most finitely many k with probability 1.

(b) If $\lambda \geq p^{-1}$ then with probability 1, the events $A_k^{(\lambda)}$ happen for infinitely many k.

Solution

(a) Suppose $\lambda < p^{-1}$. That is, $0 < \lambda p < 1$.
Notice that

$$A_k^{(\lambda)} = \bigcup_{m=\lfloor \lambda^k \rfloor}^{\lfloor \lambda^{k+1} \rfloor - k} A_m,$$

where

$$A_m = \{X_m = X_{m+1} = \cdots = X_{m+k-1} = 1\}.$$

Therefore,

$$\mathbb{P}\Big(A_k^{(\lambda)}\Big) \leq \sum_{m=\lfloor \lambda^k \rfloor}^{\lfloor \lambda^{k+1} \rfloor - k} \mathbb{P}(A_m) = \sum_{m=\lfloor \lambda^k \rfloor}^{\lfloor \lambda^{k+1} \rfloor - k} p^k \leq \lambda^{k+1} p^k = \lambda (\lambda p)^k.$$

Since $0 < \lambda p < 1$,

$$\sum_{k=1}^{\infty} \mathbb{P}\Big(A_k^{(\lambda)}\Big) \leq \lambda \sum_{k=1}^{\infty} (\lambda p)^k = \lambda \frac{\lambda p}{1 - \lambda p} < \infty.$$

Hence, by the first Borel-Cantelli lemma

$$\mathbb{P}\Big(A_k^{(\lambda)} \text{ happens for only finitely many } k\Big) = 1.$$

(b) Suppose $\lambda \geq p^{-1}$. That is, $\lambda p \geq 1$. Let

$$B_m = \{X_{\lfloor \lambda^k \rfloor + mk} = \cdots = X_{\lfloor \lambda^k \rfloor + (m+1)k - 1} = 1\}$$

for $m = 0, 1, \ldots, M_k$, where $M_k = \left\lfloor \frac{\lfloor \lambda^{k+1} \rfloor - \lfloor \lambda^k \rfloor}{k} \right\rfloor \geq \frac{\lambda^{k+1} - 1 - \lambda^k}{k} - 1$.
Notice that

$$\bigcup_{m=1}^{M_K} B_m \subseteq A_k^{(\lambda)}.$$

Moreover, since the events B_m are independent

$$\mathbb{P}\left(\bigcup_{m=1}^{M_k} B_m\right) = 1 - \mathbb{P}\left(\bigcap_{k=1}^{M_k} B_k^C\right) = 1 - \prod_{k=1}^{M_k} (1 - \mathbb{P}(B_m)) = 1 - \prod_{k=1}^{M_k} (1 - p^k) = 1 - (1 - p^k)^{M_k}.$$

Using the inequality $1 - x \leq e^{-x}$ we have

$$\mathbb{P}\left(\bigcup_{m=1}^{M_k} B_m\right) \geq 1 - e^{-M_k \cdot p^k}$$

Moreover, since $0 < p < 1$ and $\lambda p \geq 1$

$$M_k \cdot p^k \geq \lambda \frac{(\lambda p)^k}{k} - \frac{p^k}{k} - p^k \geq \lambda \frac{(\lambda p)^k}{k} - 2p^k \geq \frac{1}{k} - 2p^k.$$

Then we have

$$\mathbb{P}\left(\bigcup_{m=1}^{M_k} B_m\right) \geq 1 - \exp\left(-\frac{1}{k} + 2p^k\right).$$

Hence, we have the lower bound

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k^{(\lambda)}) \geq \sum_{k=1}^{\infty} \left(1 - \exp\left(2p^k - \frac{1}{k}\right)\right)$$

We know that

$$\sum_{k=1}^{\infty} \left(1 - \exp\left(2p^k - \frac{1}{k}\right)\right) = \infty \quad \text{if and only if} \quad \prod_{k=1}^{\infty} \exp\left(2p^k - \frac{1}{k}\right) = 0.$$

Which is exactly the case, since

$$\lim_{n\to\infty} \prod_{k=1}^{n} \exp\left(2p^k - \frac{1}{k}\right) = \lim_{n\to\infty} \exp\left(2\sum_{k=1}^{n} p^k - \sum_{k=1}^{n} \frac{1}{k}\right) = \exp(-\infty) = 0.$$

Therefore,

$$\sum_{k=1}^{\infty} \mathbb{P}\Big(A_k^{(\lambda)}\Big) = \infty.$$

Since for large enough k the intervals $[\lambda^k, \lambda^{k+1} - 1]$ are disjoint, the events $A_k^{(\lambda)}$ are independent. Therefore, by the second Borel-Cantelli lemma it follows that

$$\mathbb{P}\Big(A_k^{(\lambda)} \text{ happens for infinitely many } k\Big) = 1.$$