10th Practice Class
Kolmogorov's 0-1 Law and Iterated Logarithm

10.1 Let $X_1, X_2, \dots$ be uncorrelated random variables on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and expected value 0 respectively. Show that if the sum of the variances is finite then the sequence $S_n = X_1 + \dots + X_n$ is convergent in L^2.

Solution We show that the sequence (S_n) is Cauchy. By assumption, we know that $\sum_{j=1}^{\infty} \mathbb{E}(X_j^2) < \infty$. Let $n \le m \in \mathbb{N}$. It follows

$$\mathbb{E}((S_m - S_n)^2) = \mathbb{E}\left(\left(\sum_{j=n+1}^{m} X_j\right)^2\right) \overset{(*)}{=} \sum_{j=n+1}^{m} \mathbb{E}(X_j^2) \le \sum_{j=n+1}^{\infty} \mathbb{E}(X_j^2) \to 0 \quad \text{as } n \to \infty.$$

We used at $(*)$ that the variables are uncorrelated and have zero mean:

$$\mathbb{E}\left(\left(\sum_{j=n+1}^{m} X_j\right)^2\right) = \mathbb{D}^2\left(\sum_{j=n+1}^{m} X_j\right) = \sum_{j=n+1}^{m} \mathbb{D}^2(X_j) = \sum_{j=n+1}^{m} \mathbb{E}(X_j^2).$$

Since $L^2(\Omega, \mathcal{A}, \mathbb{P})$ is a complete metric space, there exits a random variable $S \in L^2$ such that

$$S_n \xrightarrow{L^2} S.$$

10.2 Let $X_1, X_2, \dots$ be independent random variables such that $X_n \sim \text{Exp}(\lambda_n)$ and let $S_n := \sum_{i=1}^{n} X_i$. Prove that if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$ then $S_n \xrightarrow{\text{a.s.}} \infty$, but if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$ then $S_n \xrightarrow{\text{a.s.}} S$, where S is a random variable with $\mathbb{P}(S < \infty) = 1$.

Solution

(i) Suppose $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$. Choose a subsequence $1 = n_1 < n_2 < n_3 < \dots$ such that

$$\sum_{j=n_k}^{n_{k+1}-1} \frac{1}{\lambda_j} \ge 2^k.$$

Then let us consider the blocks

$$Y_k = X_{n_k} + X_{n_k+1} + \dots + X_{n_{k+1}-1}.$$

Then using Markov's inequality and the independence

$$\mathbb{P}(Y_k \le 1) = \mathbb{P}(e^{-Y_k} \ge e^{-1}) \le e\,\mathbb{E}(e^{-Y_k}) = e \prod_{j=n_k}^{n_{k+1}-1} \mathbb{E}(e^{-X_j}),$$

where

$$\mathbb{E}(e^{-X_j}) = \int_0^{\infty} e^{-x} \lambda_j e^{-\lambda_j x}\, dx = \frac{1}{1 + \frac{1}{\lambda_j}}.$$

Hence,

$$\mathbb{P}(Y_k \le 1) \le e \prod_{j=n_k}^{n_{k+1}-1} \frac{1}{1 + \frac{1}{\lambda_j}} = e\left(\prod_{j=n_k}^{n_{k+1}-1} \left(1 + \frac{1}{\lambda_j}\right)\right)^{-1}.$$

Moreover,

$$\prod_{j=n_k}^{n_{k+1}-1} \left(1 + \frac{1}{\lambda_j}\right) \ge \sum_{j=n_k}^{n_{k+1}-1} \frac{1}{\lambda_j} \ge 2^k.$$

It follows that

$$\mathbb{P}(Y_k \le 1) \le \frac{e}{2^k}.$$

Hence, the sequence is summable:

$$\sum_{k=1}^{\infty} \mathbb{P}(Y_k \leq 1) \leq \sum_{k=1}^{\infty} \frac{e}{2^k} = e < \infty.$$

Therefore, by the first Borel-Cantelli lemma

$$\mathbb{P}(Y_k \leq 1 \text{ for infinitely many } k) = 0.$$

As a consequence,

$$S_n \xrightarrow{\text{a.s.}} \infty.$$

(ii) Suppose $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$. Then notice

$$\lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{E}(X_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{\lambda_j} \quad \text{exits} \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{D}^2(X_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{\lambda_j^2} < \infty.$$

The first convergence comes by assumption. For the second once, notice that since $1/\lambda_j \to 0$ (necessary condition for the sum to be convergent), there is a n_0 such that for $j \geq n_0$

$$0 < \frac{1}{\lambda_j} \leq 1.$$

In particular, for $j \geq n_0$

$$0 < \frac{1}{\lambda_j^2} \leq \frac{1}{\lambda_j} \leq 1.$$

Hence,

$$\sum_{j=1}^{\infty} \frac{1}{\lambda^2} \leq \sum_{j=1}^{n_0-1} \frac{1}{\lambda^2} + \sum_{j=n_0}^{\infty} \frac{1}{\lambda_j} < \infty.$$

Therefore, by Kolmogorov's two series theorem there exits a finite random variables S such that

$$S_n \xrightarrow{\text{a.s.}} S.$$

10.5 Let $X_1, X_2, \ldots$ be independent random variables and let b_n be a sequence such that $b_n \nearrow \infty$.
(a) Prove that $\mathbb{P}\Big(\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} X_i \text{ exists}\Big)$ is either 0 or 1.
(b) Suppose that $\frac{1}{b_n} \sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} Z$ for some random variable Z. Show that Z is constant almost surely.

Solution

(a) Since for any $k \geq 1$

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} X_i = \underbrace{\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{k} X_i}_{= 0 \text{ for any realization}} + \lim_{n\to\infty} \frac{1}{b_n} \sum_{i=k+1}^{n} X_i = \lim_{n\to\infty} \frac{1}{b_n} \sum_{i=k+1}^{n} X_i,$$

it is clear that

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} X_i$$

is a tail-event. Since the variables are independent, by Kolmogorov's 0-1 Law it follows that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} X_i\right) = 0 \text{ or } 1.$$

(b) Let $c \in \mathbb{R}$. Then by similar argument as above it is clear that

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} X_i = c$$

is a tail-event. Therefore,

$$\mathbb{P}(Z = c) = \mathbb{P}\left(\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} X_i = c \right) = 0 \text{ or } 1.$$

Since Z is a finite real-valued random variable, there exists a $\tilde{c} \in \mathbb{R}$ such that

$$\mathbb{P}(Z = \tilde{c}) = 1.$$

10.6 Let $X_0, X_1, X_2, \dots$ be a sequence of independent random variables. Show that the radius of convergence of the random Taylor series $\sum_{k=0}^{\infty} X_k z^k$ is r with probability 1 for some non-random $r \in \mathbb{R}_+ \cup \{0, \infty\}$.

Solution The radious of convergence is the random variable

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|X_n|}}.$$

For any $\rho > 0$ let

$$A_\rho = \{\limsup_{n\to\infty} \sqrt[n]{|X_n|} \geq \rho\}.$$

Then notice, A_ρ is a tail event. By Kolmogorov's 0-1 Law

$$\mathbb{P}(A_\rho) = 0 \qquad \text{or} \qquad \mathbb{P}(A_\rho) = 1.$$

Moreover, if $\rho_1 \leq \rho_2$, then $A_{\rho_2} \subseteq A_{\rho_1}$. In particular,

$$\mathbb{P}(A_{\rho_2}) \leq \mathbb{P}(A_{\rho_1}).$$

We have three cases.

(1) For every $\rho > 0$,

$$\mathbb{P}(A_\rho) = 1.$$

That is,

$$\mathbb{P}\left(\limsup_{n\to\infty} \sqrt[n]{|X_n|} = \infty \right) = 1.$$

It follows

$$\mathbb{P}(R = 0) = 1.$$

(2) For every $\rho > 0$,

$$\mathbb{P}(A_\rho) = 0.$$

That is,

$$\mathbb{P}\left(\limsup_{n\to\infty} \sqrt[n]{|X_n|} = 0 \right) = 1.$$

It follows

$$\mathbb{P}(R = \infty) = 1.$$

(3) There is a $\tilde{\rho} > 0$ such that for $\rho_1 < \tilde{\rho} < \rho_2$

$$\mathbb{P}(A_{\rho_1}) = 1 \quad \text{and} \quad \mathbb{P}(A_{\rho_2}) = 0.$$

That is,

$$\mathbb{P}\left(\limsup_{n\to\infty} \sqrt[n]{|X_n|} = \tilde{\rho} \right) = 1.$$

It follows

$$\mathbb{P}(R = 1/\tilde{\rho}) = 1.$$