11th Practice Class
Characteristic functions I

11.1 (a) Show that $\phi_{aX+b}(u) = e^{ibu}\phi_X(au)$.

(b) Show if X, Y are independent the characteristic function ϕ_{X+Y} of $X+Y$ is the product of the characteristic functions ϕ_X, ϕ_Y.

(c) Show an example that $\phi_{X+Y} = \phi_X\phi_Y$ but X and Y are not independent.

Solution

(a)
$$\phi_{aX+b}(u) = \mathbb{E}(e^{iu(aX+b)}) = \mathbb{E}(e^{iub}e^{i(au)X}) = e^{ibu}\,\mathbb{E}(e^{i(au)X}) = e^{ibu}\phi_X(au).$$

(b) Since X, Y are independent, e^{iuX} and e^{iuY} are also independent. It follows
$$\phi_{X+Y}(u) = \mathbb{E}(e^{iu(X+Y)}) = \mathbb{E}(e^{iuX}e^{iuX}) = \mathbb{E}(e^{iuX})\,\mathbb{E}(e^{iuY}) = \phi_X(u)\phi_Y(u).$$

(c) Let $X = Y \sim \mathrm{Cau}(0,1)$. Then $X+Y = 2X \sim \mathrm{Cau}(0,2)$. The characteristic functions are
$$\phi_X(u) = \phi_Y(u) = e^{-|u|} \;\Rightarrow\; \phi_X(u)\phi_Y(u) = e^{-|u|}e^{-|u|} = e^{-2|u|},$$
$$\phi_{X+Y}(u) = \phi_{2X}(u) = \phi_X(2u) = e^{-|2u|} = e^{-2|u|}.$$
However, clearly X and Y are not independent.

11.2 Find the characteristic functions of the following distributions: $\mathrm{Bin}(n,p)$, $\mathrm{Geo}(p)$, $\mathrm{Poi}(\lambda)$, $\mathrm{Exp}(\lambda)$, $\mathcal{N}(\mu,\sigma^2)$, $\mathrm{Cau}(a,b)$ and $\mathrm{Uni}(a,b)$.

Solution

- Let $X \sim \mathrm{Bin}(n,p)$.
$$\phi_X(u) = \sum_{k=0}^{n} e^{iuk}\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=0}^{n}\binom{n}{k}(pe^{iu})^k(1-p)^{n-k} = (1-p+pe^{iu})^n.$$
Different solution. Notice that
$$X \stackrel{d}{=} Y_1 + \dots Y_n,$$
where $Y_1, \dots, Y_n$ are i.i.d with $\mathbb{P}(Y_k = 1) = 1 - \mathbb{P}(Y_k = 0) = p$. Then the characteristic function of Y_1 is
$$\phi_{Y_1}(u) = (1-p)e^{iu\cdot 0} + pe^{iu\cdot 1} = 1 - p + pe^{iu}.$$
It follows that the characteristic function of X is
$$\phi_X(u) = \phi_{Y_1}(u)^n(1-p+pe^{iu})^n.$$

- Let $X \sim \mathrm{Exp}(1)$. Then
$$\phi_X(u) = \int_0^\infty e^{iux}e^{-x}\,\mathrm{d}x = \int_0^\infty e^{x(iu-1)}\,\mathrm{d}x = \left[\frac{e^{x(iu-1)}}{ui-1}\right]_{x=0}^{x\to\infty} = \frac{-1}{ui-1} + \underbrace{\lim_{x\to\infty}\frac{e^{x(iu-1)}}{ui-1}}_{=0},$$
since
$$\left|e^{x(iu-1)}\right| = \underbrace{\left|e^{iux}\right|}_{=1}\cdot\left|e^{-x}\right| = e^{-x} \to 0 \quad \text{as } x\to\infty.$$
Hence,
$$\phi_X(u) = \frac{-1}{iu-1} = \frac{1}{1-iu}.$$
Now, let $Y \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$. Then notice that
$$Y \stackrel{d}{=} \frac{1}{\lambda}X.$$
Therefore, its characteristic function is
$$\phi_Y(u) = \phi_{\frac{1}{\lambda}X}(u) = \phi_X\left(\frac{u}{\lambda}\right) = \frac{1}{1-\frac{iu}{\lambda}} = \frac{\lambda}{\lambda - iu}.$$

- Let $X \sim \mathcal{N}(0,1)$. We know that its moment generating function is

$$Z(u) = \mathbb{E}(e^{uX}) = \exp\left(\frac{u^2}{2}\right).$$

For X, all absolute moments are finite and $u \mapsto Z(u)$ is analytic on $\mathbb{R}$. In that case, there is a unique analytic extension to $\mathbb{C}$. Therefore, we can express the characteristic function as

$$\phi_X(u) = Z(iu) = \exp\left(\frac{(iu)^2}{2}\right) = \exp\left(-\frac{u^2}{2}\right).$$

Now, let $Y \sim \mathcal{N}(\mu, \sigma^2)$. Then notice that

$$Y \stackrel{d}{=} \mu + \sigma X.$$

Therefore, its characteristic function is

$$\phi_Y(u) = \phi_{\mu+\sigma X}(u) = e^{iu\mu}\phi_X(\sigma u) = \exp(iu\mu)\exp\left(-\frac{(\sigma u)^2}{2}\right) = \exp\left(iu\mu - \frac{\sigma^2 u^2}{2}\right).$$

Different solution. With direct complex integration. Stay tuned.

11.3 (a) Let ϕ be a characteristic function. Show that $\overline{\phi}, \phi^2, |\phi|^2$ are characteristic functions too.

(b) Let $\phi_1, \ldots, \phi_n$ be characteristic functions. Show that $\sum_{i=1}^n q_i \phi_i$ is a characteristic function for any $q_1, \ldots, q_n \geq 0$ with $\sum_{i=1}^n q_i = 1$. In fact, show that $\mathrm{Re}(\phi)$ is a characteristic function too.

(c) Show an example that $|\phi|$ is not a characteristic function.

Solution

(a) Let X has characteristic function ϕ.

- Let $Y = -X$. Then Y has characteristic function by linearity of the expectation

$$\mathbb{E}(e^{iuY}) = \mathbb{E}(e^{-iuX}) = \mathbb{E}(\overline{e^{iuX}}) = \overline{\mathbb{E}(e^{iuX})} = \overline{\phi(u)}.$$

- Let Y_1, Y_2 be i.i.d having the same distribution as X. Then the characteristic function of $Y_1 + Y_2$ is

$$\mathbb{E}(e^{iu(Y_1+Y_2)}) = \mathbb{E}(e^{iuY_1})\,\mathbb{E}(e^{iuY_2}) = \phi(u)\phi(u) = \phi(u)^2.$$

- Let Y_1, Y_2 be i.i.d having the same distribution as X. Then the characteristic function of $Y_1 - Y_2$ is

$$\mathbb{E}(e^{iu(Y_1-Y_2)}) = \mathbb{E}(iuY_1)\,\mathbb{E}(-iuY_2) = \phi(u)\overline{\phi(u)} = |\phi(u)|^2.$$

(b) Let $X_1, \ldots, X_n$ be independent with characteristic functions $\phi_1, \ldots, \phi_n$. Let Y be independent of $X_1, \ldots, X_n$ with distribution

$$\mathbb{P}(Y = k) = q_k \quad \text{for } k = 1, \ldots, n.$$

Consider the variable

$$Z = X_Y.$$

Then Z has characteristic function

$$\mathbb{E}(e^{iuZ}) = \mathbb{E}(e^{iuX_Y}) = \sum_{k=1}^n \mathbb{P}(Y=k)\,\mathbb{E}(e^{iuX_Y} \mid Y = k) = \sum_{k=1}^n q_k\,\mathbb{E}(e^{iuX_k}) = \sum_{k=1}^n q_k \phi_k(u).$$

In particular, let X_1 have characteristic function ϕ. Let X_2 be independent of X_1 with the same distibution as $-X_1$. Moreover, let $\mathbb{P}(Y = 1) = \mathbb{P}(Y = 2) = 1/2$. Then Z has characteristic function

$$\frac{1}{2}\phi(u) + \frac{1}{2}\overline{\phi(u)} = \mathrm{Re}(\phi(u)).$$

11.5 Are the following functions characteristic function for some random variable?

$$\text{(a) } \frac{1}{1+u^2}, \quad \text{(b) } e^{-u^4}, \quad \text{(c) } \sin u, \quad \text{(d) } \cos u, \quad \text{(e) } \frac{1+\cos u}{2}.$$

Solution

(a) Let X_1, X_2 be i.i.d with distribution Exp(1). Then $X_1 - X_2$ has characteristic function

$$\left|\frac{1}{1-iu}\right|^2 = \frac{1}{1+u^2}.$$

(b) Suppose X has characteristic function $\phi(u) = e^{-u^4}$. Since $e^{-u^4} \in C^2$ and 2 is an even number, the existence of the 2nd derivative guarantees the existence of the 2nd moment. Moreover,

$$\phi^{(2)}(0) = i^2\,\mathbb{E}(X^2) = -\,\mathbb{E}(X^2).$$

Then

$$\phi^{(2)}(0) = \frac{\mathrm{d}^2}{\mathrm{d}u^2} e^{-u^4}\bigg|_{u=0} = 16u^6 e^{-u^4} - 12u^2 e^{-u^4}\Big|_{u=0} = 0.$$

Hence,

$$\mathbb{E}(X^2) = 0.$$

It can happen if and only if

$$\mathbb{P}(X=0) = 1.$$

However, it that case

$$\phi(u) = \mathbb{E}(e^{iuX}) = 1 \cdot e^{iu \cdot 0} = 1 \neq e^{-u^4}.$$

Therefore, it is not a characteristic function.

(c) Since $\sin(0) = 0 \neq 1$, it is not a characteristic function.

(d) Let X be uniform on the set $\{-1, 1\}$. Then it has characteristic function

$$\frac{1}{2}e^{-iu} + \frac{1}{2}e^{iu} = \cos(u).$$

(e) Let X has distribution

$$\mathbb{P}(X=-1) = \mathbb{P}(X=1) = \frac{1}{4},\ \mathbb{P}(X=0) = \frac{1}{2}.$$

Then X has characteristic function

$$\frac{1}{2}e^{0} + \frac{1}{4}e^{-iu} + \frac{1}{4}e^{iu} = \frac{1}{2} + \frac{1}{2}\cos(u) = \frac{1+\cos(u)}{2}.$$