

12th Exercise Sheet

Characteristic functions II

- 12.1** (a) Let U be a random variable with distribution $\text{Uni}(0, 1)$, and let Y and Z be independent and independent of U with distribution $\text{Exp}(1)$. Show that $X = U(Y + Z)$ has distribution $\text{Exp}(1)$ as well.
- (b) For which distributions F does it hold that if Y and Z are independent random variables with distribution F , and U is independent of Y and Z with distribution $\text{Uni}(0, 1)$ then $X = U(Y + Z)$ has distribution F ?

HW 12.2 Let X and Y be independent random variables with distribution $\mathcal{N}(0, 1)$, moreover, let U, V be independent and independent of X and Y with distribution $\text{Uni}(0, 1)$. Show that the distribution of $\frac{UX+VY}{\sqrt{U^2+V^2}}$ is standard normal.

- 12.3** Let Y be a random variable with distribution $N(0, 1)$. Show that the characteristic function $\phi(t)$ of $X := |Y|$ is not real for every $t \in \mathbb{R} \setminus \{0\}$.

- 12.4** Let X be the log-normal distribution, that is, $X = e^Y$, where $Y \sim N(0, 1)$.

- (a) Show that the moments of X are $e^{k^2/2}$, and find its density function $f_X(x)$.
- (b) Show that the function $g_{n,a}(x) := (1 + a \sin(2\pi n \log(x)))f_X(x)$ is a density function for every $a \in (-1, 1)$ and $n \in \mathbb{N}$.
- (c) Show that the moments of $g_{n,a}$ are $e^{k^2/2}$ for every $a \in (-1, 1)$ and $n \in \mathbb{N}$.

- 12.5** (a) Show by using a probabilistic approach that $\int_{-\infty}^{\infty} \frac{\sin^2(t)}{t^2} dt = \pi$.

(b) Show that $\frac{1}{\pi} \int_0^{\infty} \frac{\sin(at)}{t} dt = \begin{cases} \frac{1}{2} & \text{if } a > 0, \\ -\frac{1}{2} & \text{if } a < 0, \\ 0 & \text{if } a = 0. \end{cases}$

- (c) (*Inversion formula II*) Let ϕ be the characteristic function of the random variable X with distribution function F . Then show that

$$\mathbb{P}(a < X < b) + \frac{\mathbb{P}(X = a) + \mathbb{P}(X = b)}{2} = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi(t) dt.$$

HW 12.6 Show that if $X_1, X_2, \dots, X_n$ are independent and uniformly distributed on $(-1, 1)$ then for $n \geq 2$ the random variable $X_1 + \dots + X_n$ has density

$$f(x) = \frac{1}{\pi} \int_0^{\infty} (\sin(t)/t)^n \cos(tx) dt.$$

- 12.7** (a) Let X be a random variable with characteristic function ϕ . Show that

$$\mathbb{P}(X = a) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-ita} \phi(t) dt.$$

- (b) Find the value of $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\phi(t)|^2 dt$.

HW₂ 12.8

- (a) Let X be a random variable with characteristic function ϕ . Suppose that $\mathbb{P}(X \in h\mathbb{Z}) = 1$, with some $h > 0$. Show that

$$\mathbb{P}(X = a) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-ita} \phi(t) dt \text{ for } a \in h\mathbb{Z}.$$

- (b) Suppose now that $\mathbb{P}(Y \in b + h\mathbb{Z}) = 1$ with $b \in [0, h)$. Using part (a), show that the previous formula holds for Y with $a \in b + h\mathbb{Z}$. (Hint: Use exercise [12.7a](#))

12.9 Let X and Y be i.i.d random variables with expected value 0 and variance 1. Denote the common characteristic function by ϕ . Suppose that $X+Y$ and $X-Y$ are independent. Show that this is possible only if X and Y are standard normal random variables.

HW12.10 *Invertability of the Laplace-transformation.* Let $f: [0, \infty) \mapsto \mathbb{R}$ be a continuous and bounded map. We define the Laplace-transform $\hat{f}$ of f as

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda y} f(y) dy.$$

- (a) Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution $\text{Exp}(\lambda)$, i.e. $\mathbb{P}(X_k > x_k) = e^{-\lambda x}$; $x > 0$, $\mathbb{E}(X_k) = \lambda^{-1}$, and $\mathbb{D}^2(X_k) = \lambda^{-2}$. Show that

$$\mathbb{E}(f(S_n)) = (-1)^{n-1} \frac{\lambda^n \hat{f}^{(n-1)}(\lambda)}{(n-1)!},$$

where $S_n = X_1 + \dots + X_n$ and $\hat{f}^{(n-1)}$ denotes the $n-1$ th derivative of $\hat{f}$.

- (b) Prove that the Laplace transform satisfies the following inversion formula (for all continuous and bounded map f):

$$f(y) = \lim_{n \rightarrow \infty} (-1)^{n-1} \frac{(n/y)^n \hat{f}^{(n-1)}(n/y)}{(n-1)!}.$$