12th Practice Class
Characteristic functions II

12.1 (a) Let U be a random variable with distribution $\mathrm{Uni}(0,1)$, and let Y and Z be independent and independent of U with distribution $\mathrm{Exp}(1)$. Show that $X = U(Y+Z)$ has distribution $\mathrm{Exp}(1)$ as well.

(b) For which distributions F does it hold that if Y and Z are independent random variables with distribution F, and U is independent of Y and Z with distribution $\mathrm{Uni}(0,1)$ then $X = U(Y+Z)$ has distribution F?

Solution

(a) The characteristic function of $Y+Z$ is

$$\phi_{Y+Z}(t) = \frac{1}{(1-it)^2}.$$

Then characteristic function at $t \neq 0$ of $U(Y+Z)$ is

$$\begin{aligned}\mathbb{E}(e^{itU(Y+Z)}) &= \mathbb{E}\Big(\mathbb{E}\Big(e^{itU(Y+Z)} \;\Big|\; U\Big)\Big) = \mathbb{E}\Big(\frac{1}{(1-itU)^2}\Big) \\ &= \int_0^1 \frac{1}{(1-itu)^2}\,\mathrm{d}u = \Big[\frac{1}{it}\cdot\frac{1}{1-itu}\Big]_{u=0}^{u=1} = \frac{1}{it}\cdot\frac{1}{1-it} - \frac{1}{it} = \frac{1-(1-it)}{it(1-it)} = \frac{1}{1-it}.\end{aligned}$$

For $t=0$ it is clearly 1 and

$$1 = \frac{1}{1-i\cdot 0}.$$

Hence, the characteristic function of $U(Y+Z)$ for all $t \in R$ is

$$t \mapsto \frac{1}{1-it},$$

which is the characteristic function of $\mathrm{Exp}(1)$. Therefore, $U(Y+Z) \sim \mathrm{Exp}(1)$.

12.5 (a) Show by using a probabilistic approach that $\int_{-\infty}^{\infty} \frac{\sin^2(t)}{t^2}dt = \pi$.

(b) Show that $\frac{1}{\pi}\int_0^{\infty} \frac{\sin(at)}{t}dt = \begin{cases} \frac{1}{2} & \text{if } a>0, \\ -\frac{1}{2} & \text{if } a<0, \\ 0 & \text{if } a=0. \end{cases}$

(c) *(Inversion formula II)* Let ϕ be the characteristic function of the random variable X with distribution function F. Then show that

$$\mathbb{P}(a<X<b) + \frac{\mathbb{P}(X=a)+\mathbb{P}(X=b)}{2} = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi(t)dt.$$

Solution

(a) If ϕ is the characteristic function of $X \sim \mathrm{Uni}(-1,1)$, then

$$\phi(t) = \frac{\sin(t)}{t}.$$

Notice that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\sin^2(t)}{t^2}\,\mathrm{d}t &= \int_{-\infty}^{\infty}\frac{e^{it}-e^{-it}}{2it}\frac{\sin(t)}{t}\,\mathrm{d}t \\ &= \pi\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{it}-e^{-it}}{it}\frac{\sin(t)}{t}\,\mathrm{d}t = \pi\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{it}-e^{-it}}{it}\phi(t)\,\mathrm{d}t.\end{aligned}$$

Since integral is absolute convergent and -1 and 1 are points of continuity of the distribution X, by the inversion formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{it}-e^{-it}}{it}\phi(t)\,\mathrm{d}t = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{it}-e^{-it}}{it}\phi(t)\,\mathrm{d}t = \mathbb{P}(X<1)-\mathbb{P}(X<-1) = 1-0 = 1.$$

It follows that

$$\int_{-\infty}^{\infty}\frac{\sin^2(t)}{t^2}\,\mathrm{d}t = \pi\cdot 1 = \pi.$$

(b) Let $X=0$ almost surely. Then X has characteristic function $\phi(t)=1$. Then by the inversion formula

$$\begin{aligned}\mathbb{P}(X<a)-\mathbb{P}(X<-a) &= \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{iat}-e^{-iat}}{it}\,\mathrm{d}t = \lim_{T\to\infty}\frac{1}{\pi}\int_{-T}^{T}\frac{\sin(at)}{t}\,\mathrm{d}t \\ &= \lim_{T\to\infty}\frac{1}{\pi}\int_{-T}^{T}\frac{\sin(at)}{t}\,\mathrm{d}t = 2\cdot\lim_{T\to\infty}\frac{1}{\pi}\int_{0}^{T}\frac{\sin(at)}{t}\,\mathrm{d}t = 2\cdot\frac{1}{\pi}\int_{0}^{\infty}\frac{\sin(at)}{t}\,\mathrm{d}t\end{aligned}$$

Moreover, the left hand side

$$\mathbb{P}(X<a)-\mathbb{P}(X<-a) = \begin{cases}1 & \text{if } a>0,\\ -1 & \text{if } a<0,\\ 0 & \text{if } a=0.\end{cases}$$

Remark. The integral is not absolute convergent for $a\neq 0$.

(c) By the definition of the characteristic function

$$\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-iat}-e^{-ibt}}{it}\phi(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-iat}-e^{-ibt}}{it}\,\mathbb{E}(e^{itX})\,\mathrm{d}t = \frac{1}{2\pi}\int_{-T}^{T}\mathbb{E}\left(\frac{e^{-iat}-e^{-ibt}}{it}e^{itX}\right)\mathrm{d}t.$$

Since $\left|e^{ir}-e^{is}\right|\leq|r-s|$, the next upper bound holds:

$$\frac{\left|e^{-iat}-e^{-ibt}\right|}{|it|}\cdot\left|e^{itX}\right| = \frac{\left|e^{-iat}-e^{-ibt}\right|}{|t|}\cdot 1 \leq \frac{|at-bt|}{|t|} = |a-b|\,.$$

Therefore, using Fubini's theorem the expression equals to

$$\mathbb{E}\left(\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{it(X-a)}-e^{it(X-b)}}{it}\,\mathrm{d}t\right).$$

Using Euler's formula $e^{ix}=\cos(x)+i\sin(x)$, the real part $\cos(x)/t$ is an odd function, so its integral from $-T$ to T vanishes. Therefore,

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{2\pi}\int_{-T}^{T}\frac{i\sin(t(X-a))-i\sin(t(X-b))}{it}\,\mathrm{d}t\right)\\ &\qquad = \mathbb{E}\left(\frac{1}{\pi}\int_{0}^{T}\frac{\sin(t(X-a))}{t}\,\mathrm{d}t-\frac{1}{\pi}\int_{0}^{T}\frac{\sin(t(X-b))}{t}\,\mathrm{d}t\right).\end{aligned}$$

Using the convergence in part b) there exists a $T_0>0$ such that for all $T\geq T_0$

$$\left|\frac{1}{\pi}\int_{0}^{T}\frac{\sin(t(X-a))}{t}\,\mathrm{d}t-\frac{1}{\pi}\int_{0}^{T}\frac{\sin(t(X-b))}{t}\,\mathrm{d}t\right|\leq 2.$$

Hence, by the Dominated convergence theorem

$$\begin{aligned}&\lim_{T\to\infty}\mathbb{E}\left(\frac{1}{\pi}\int_{0}^{T}\frac{\sin(t(X-a))}{t}\,\mathrm{d}t-\frac{1}{\pi}\int_{0}^{T}\frac{\sin(t(X-b))}{t}\,\mathrm{d}t\right)\\ &\qquad = \mathbb{E}\left(\lim_{T\to\infty}\frac{1}{\pi}\int_{0}^{T}\frac{\sin(t(X-a))}{t}\,\mathrm{d}t-\frac{1}{\pi}\int_{0}^{T}\frac{\sin(t(X-b))}{t}\,\mathrm{d}t\right).\end{aligned}$$

Moreover, using the convergence in part b) again

$$\lim_{T\to\infty}\frac{1}{\pi}\int_0^T\frac{\sin(t(X-a))}{t}\,\mathrm{d}t-\frac{1}{\pi}\int_0^T\frac{\sin(t(X-b))}{t}\,\mathrm{d}t$$
$$=0\cdot\mathbb{1}\{X<0\}+\frac{1}{2}\cdot\mathbb{1}\{X=a\}+1\cdot\mathbb{1}\{a<X<b\}+\frac{1}{2}\cdot\mathbb{1}\{X=b\}+0\cdot\mathbb{1}\{b<X\}.$$

This exactly means that the random variable inside the expectation is

$$\mathbb{1}\{a<X<b\}+\frac{1}{2}\mathbb{1}\{X=a\}+\frac{1}{2}\mathbb{1}\{X=b\}.$$

Taking the expected value yields the final result:

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^T\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\,\mathrm{d}t=\mathbb{E}\Big(\mathbb{1}\{a<X<b\}+\frac{1}{2}\mathbb{1}\{X=a\}+\frac{1}{2}\mathbb{1}\{X=b\}\Big)$$
$$=\mathbb{P}(a<X<b)+\frac{\mathbb{P}(X=a)+\mathbb{P}(X=b)}{2}.$$

12.7 (a) Let X be a random variable with characteristic function ϕ. Show that

$$\mathbb{P}(X=a)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^T e^{-ita}\phi(t)dt.$$

(b) Find the value of $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^T|\phi(t)|^2\,dt$.

Solution

(a) By the definition of the characteristic function

$$\frac{1}{2T}\int_{-T}^T e^{-iat}\phi(t)\,\mathrm{d}t=\frac{1}{2T}\int_{-T}^T e^{-iat}\,\mathbb{E}\Big(e^{itX}\Big)\,\mathrm{d}t=\frac{1}{2T}\int_{-T}^T\mathbb{E}\Big(e^{it(X-a)}\Big)\,\mathrm{d}t$$
$$=\frac{1}{2T}\int_{-T}^T\mathbb{E}\Big(e^{it(X-a)}\mathbb{1}\{X=a\}\Big)\,\mathrm{d}t+\frac{1}{2T}\int_{-T}^T\mathbb{E}\Big(e^{it(X-a)}\mathbb{1}\{X\neq a\}\Big)\,\mathrm{d}t$$
$$=\frac{1}{2T}\int_{-T}^T\mathbb{E}(\mathbb{1}\{X=a\})\,\mathrm{d}t+\frac{1}{2T}\int_{-T}^T\mathbb{E}\Big(e^{it(X-a)}\mathbb{1}\{X\neq a\}\Big)\,\mathrm{d}t=$$
$$\mathbb{P}(X=a)+\frac{1}{2T}\int_{-T}^T\mathbb{E}\Big(e^{it(X-a)}\mathbb{1}\{X\neq a\}\Big)\,\mathrm{d}t.$$

Since $\left|e^{-it(X-a)}\right|\le 1$, using Fubini's theorem

$$\frac{1}{2T}\int_{-T}^T\mathbb{E}\Big(e^{it(X-a)}\mathbb{1}\{X\neq a\}\Big)\,\mathrm{d}t=\mathbb{E}\left(\frac{1}{2T}\int_{-T}^T e^{it(X-a)}\,\mathrm{d}t\cdot\mathbb{1}\{X\neq a\}\right)$$
$$=\mathbb{E}\left(\frac{1}{2T}\int_{-T}^T\cos(t(X-a))\,\mathrm{d}t\cdot\mathbb{1}\{X\neq a\}\right)+\mathbb{E}\left(\frac{i}{2T}\int_{-T}^T\sin(t(X-a))\,\mathrm{d}t\cdot\mathbb{1}\{X\neq a\}\right)$$
$$=\mathbb{E}\left(\frac{1}{2T}\int_{-T}^T\cos(t(X-a))\,\mathrm{d}t\cdot\mathbb{1}\{X\neq a\}\right)$$

By calculating the integral we get

$$\mathbb{E}\left(\frac{1}{2T}\int_{-T}^T\cos(t(X-a))\,\mathrm{d}t\cdot\mathbb{1}\{X\neq a\}\right)=\mathbb{E}\left(\frac{1}{2T}\left[\frac{\sin(t(X-a))}{X-a}\right]_{-T}^T\cdot\mathbb{1}\{X\neq a\}\right)$$
$$=\mathbb{E}\Big(\frac{2\sin(T(X-a))}{2T(X-a)}\cdot\mathbb{1}\{X\neq a\}\Big)=\mathbb{E}\Big(\frac{\sin(T(X-a))}{T(X-a)}\cdot\mathbb{1}\{X\neq a\}\Big).$$

Notice that

$$\left|\frac{\sin(T(X-a))}{T(X-a)}\cdot\mathbb{1}\{X\neq a\}\right|\le 1$$

Therefore, by dominated convergence

$$\lim_{T\to\infty} \mathbb{E}\bigg(\frac{\sin(T(X-a))}{T(X-a)} \cdot \mathbb{1}\{X \neq a\}\bigg) = \mathbb{E}\bigg(\lim_{T\to\infty} \frac{\sin(T(X-a))}{T(X-a)} \cdot \mathbb{1}\{X \neq a\}\bigg)$$
$$= \mathbb{E}(0 \cdot \mathbb{1}\{X \neq a\}) = 0.$$

Hence,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\phi(t)\,\mathrm{d}t = \mathbb{P}(X = a) + 0 = \mathbb{P}(X = a).$$

(b) Let $\tilde{X}$ be independent from X with the same distribution. Then $X - \tilde{X}$ has characteristic function

$$|\phi(t)|^2.$$

Applying the formula in part a) with $a = 0$ to $X - \tilde{X}$

$$\mathbb{P}(X - \tilde{X} = 0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{it\cdot 0} |\phi(t)|^2 \,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t)|^2 \,\mathrm{d}t.$$

Moreover, if F is the distribution function of X, then $-\tilde{X}$ has distribution function $1 - F(-x)$. Then $X - \tilde{X}$ has distribution function

$$G(x) = \int_{-\infty}^{\infty} (1 - F(y - x))\,\mathrm{d}F(y)$$

Then by dominated convergence

$$\mathbb{P}(X - \tilde{X} = 0) = \lim_{x\to 0_+} G(x) - G(0) = \lim_{x\to 0+} \int_{-\infty}^{\infty} (F(y) - F(y - x))\,\mathrm{d}F(y)$$
$$= \int_{-\infty}^{\infty} \lim_{x\to 0_+} (F(y) - F(y - x))\,\mathrm{d}F(y) = \int_{-\infty}^{\infty} \mathbb{P}(X = y)\,\mathrm{d}F(y) = \sum_{x\in\mathbb{R}} \mathbb{P}(X = x)^2.$$

Since a random variables can only have countable many points of discontinuity, the expression makes sense.

12.9 Let X and Y be i.i.d random variables with expected value 0 and variance 1. Denote the common characteristic function by ϕ. Suppose that $X+Y$ and $X-Y$ are independent. Show that this is possible only if X and Y are standard normal random variables.

Solution Notice, that

$$2X = (X + Y) + (X - Y).$$

Since $X + Y$ and $X - Y$ are independent, taking the characteristic function of both sides yields

$$\phi(2u) = \phi(u)^2 \cdot |\phi(u)|^2.$$

Hence, for every $u \in \mathbb{R}$

$$\phi(u) = \phi\Big(\frac{u}{2}\Big)^2 \cdot \Big|\phi\Big(\frac{u}{2}\Big)\Big|^2.$$

We now prove by induction that

$$\phi(u) = \phi\Big(\frac{u}{2^n}\Big)^{2^n} \cdot \Big|\phi\Big(\frac{u}{2^n}\Big)\Big|^{4^n - 2^n}.$$

For $n = 1$ it clearly holds. Suppose it is true for some $n \geq 1$. Then using the induction formula and then our first result

$$\phi(u) = \phi\Big(\frac{u}{2^n}\Big)^{2^n} \cdot \Big|\phi\Big(\frac{u}{2^n}\Big)\Big|^{4^n - 2^n}$$
$$= \bigg(\phi\Big(\frac{1}{2}\frac{u}{2^n}\Big)^2 \Big|\phi\Big(\frac{1}{2}\frac{u}{2^n}\Big)\Big|^2\bigg)^{2^n} \cdot \bigg|\phi\Big(\frac{1}{2}\frac{u}{2^n}\Big)^2 \Big|\phi\Big(\frac{1}{2}\frac{u}{2^n}\Big)\Big|^2\bigg|^{4^n - 2^n}$$
$$= \phi\Big(\frac{u}{2^{n+1}}\Big)^{2^{n+1}} \cdot \Big|\phi\Big(\frac{u}{2^{n+1}}\Big)\Big|^{2^{n+1} + 4^{n+1} - 2\cdot 2^{n+1}}$$
$$= \phi\Big(\frac{u}{2^{n+1}}\Big)^{2^{n+1}} \cdot \Big|\phi\Big(\frac{u}{2^{n+1}}\Big)\Big|^{4^{n+1} - 2^{n+1}}.$$

Therefore, it is true for $n+1$.

Since $\mathbb{E}(X)=0$ and $\mathbb{E}(X^2)=1$, as $u\to 0$ we have

$$\phi(u) = 1 - \frac{u^2}{2} + o(u^2).$$

Hence, as $n\to\infty$

$$\phi(u) = \left(1 - \frac{u^2}{2\cdot 4^n} + o\left(\frac{1}{4^n}\right)\right)^{2^n} \cdot \left|\left(1 - \frac{u^2}{2\cdot 4^n} + o\left(\frac{1}{4^n}\right)\right)^{4^n-2^n}\right|.$$

Since

$$\lim_{n\to\infty} 2^n\left(-\frac{u^2}{2\cdot 4^n} + o\left(\frac{1}{4^n}\right)\right) = 0,$$

for the first term we have

$$\lim_{n\to\infty} \left(1 - \frac{u^2}{2\cdot 4^n} + o\left(\frac{1}{4^n}\right)\right)^{2^n} = e^0 = 1.$$

Moreover, since

$$\lim_{n\to\infty} (4^n - 2^n)\left(-\frac{u^2}{2\cdot 4^n} + o\left(\frac{1}{4^n}\right)\right) = -\frac{u^2}{2},$$

for the second term we have

$$\lim_{n\to\infty} \left(1 - \frac{u^2}{2\cdot 4^n} + o\left(\frac{1}{4^n}\right)\right)^{4^n-2^n} = \exp\left(-\frac{u^2}{2}\right).$$

It follows

$$\begin{aligned}
\phi(u) &= \lim_{n\to\infty} \left(1 - \frac{u^2}{2\cdot 4^n} + o\left(\frac{1}{4^n}\right)\right)^{2^n} \cdot \left|\left(1 - \frac{u^2}{2\cdot 4^n} + o\left(\frac{1}{4^n}\right)\right)^{4^n-2^n}\right| \\
&= 1 \cdot \left|\exp\left(-\frac{u^2}{2}\right)\right| \\
&= \exp\left(-\frac{u^2}{2}\right).
\end{aligned}$$

This exactly means that X and Y have standard normal distribution.