13th Exercise Sheet

Weak convergence of probability distributions

13.1 Let $S = [0,1]$ and let μ_n be the discrete measure such that every point k/n has weight $1/(n+1)$ for $k = 0, 1, \dots, n$. Show that $\mu_n \Rightarrow \mu$, where μ is the Lebesgue measure on S. (Use the definition of the weak convergence, not the equivalent formalisations for distribution functions.)

13.2 Using Portmanteau Theorem, show that if X_n has distribution $\mathrm{Bin}(n, p_n)$ such that $p_n \to 0$, and $np_n \to \lambda$ as $n \to \infty$, then $X_n \xrightarrow{d} Y$, where Y has distribution $\mathrm{Poi}(\lambda)$.

HW 13.3 Let X_n be a random variable with distribution $\mathrm{Geo}(\frac{\lambda}{n})$ (that is, $\mathbb{P}(X_n = k) = (\lambda/n)(1-\lambda/n)^k$, $k \geq 0$.). Using the definition of the weak convergence, show that $\frac{X_n}{n}$ has limiting distribution and find it.

13.4 Let $c \in \mathbb{R}$ and X_n be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the following are equivalent.

(i) $X_n \xrightarrow{\mathbb{P}} c$ as $n \to \infty$.

(ii) $X_n \xrightarrow{d} c$ as $n \to \infty$.

13.5 *(Cramer-Slutsky theorem)* Let X_n, Y_n, Z_n, $n \in \mathbb{N}$ and X be random variables over the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{\mathbb{P}} y$, $Z_n \xrightarrow{\mathbb{P}} z$ for some $y, z \in \mathbb{R}$ then $X_n Y_n + Z_n \xrightarrow{d} X \cdot y + z$.

HW* 13.6 Let $F_n, F : \mathbb{R} \to [0,1]$ be cumulative distribution functions. Assume that $F_n \Rightarrow F$ and F is continuous. Show that for any convergent sequence $x_n \to x$ of real numbers we have $F_n(x_n) \to F(x)$.

13.7 *(Limit theorems for maximum of i.i.d random variables)* Let $X_1, X_2, \dots$ be i.i.d random variables with distribution function $F(x) := \mathbb{P}(X_i < x)$. Let $M_n := \max\{X_1, X_2, \dots, X_n\}$. The asymptotic behavior of the random variable M_n as $n \to \infty$ depends on then asymptotic of the upper tail of $F(x)$. Show the following limit theorems:

(a) Suppose that $F(x) < 1$ for every $x < \infty$ and $\lim_{x\to\infty} x^\alpha(1 - F(x)) = b$ for some $\alpha, b \in (0, \infty)$ Show that the distribution of $n^{-1/\alpha} M_n$ converges weakly to:

$$\mathbb{P}\Big(n^{-1/\alpha} M_n < x\Big) \to \mathbb{1}_{\{x>0\}} \exp\left(-bx^{-\alpha}\right).$$

HW* (b) Suppose that $F(x_0) = 1$ and $F(x) < 1$ for every $x < x_0$, moreover, $\lim_{x\to x_0}(x_0 - x)^{-\alpha}(1 - F(x)) = b$ for some $\alpha, b \in (0, \infty)$ Then the distribution of $n^{1/\alpha}(x_0 - M_n)$ converges weakly to:

$$\mathbb{P}\Big(n^{1/\alpha}(x_0 - M_n) < x\Big) \to \mathbb{1}_{\{x>0\}} \left(1 - \exp\left(-bx^{\alpha}\right)\right).$$

HW* (c) Suppose that $F(x) < 1$ for every $x < \infty$ and $\lim_{x\to\infty} e^{\lambda x}(1 - F(x)) = b$ for some $\lambda, b \in (0, \infty)$ Then the distribution of $M_n - \lambda^{-1}\log n$ converges weakly to:

$$\mathbb{P}\Big(M_n - \lambda^{-1}\log n < x\Big) \to \exp\Big(-be^{-\lambda x}\Big).$$

13.8 Let $X_1, X_2, \dots$ be i.i.d. random variables with standard normal distribution. From previous exercise **6.8** we know that

$$\mathbb{P}(X_i > x) \sim \frac{1}{x} e^{-x^2/2} \text{ as } x \to \infty.$$

(a) Show that for any real number θ

$$\frac{\mathbb{P}(X_i > x + \theta/x)}{\mathbb{P}(X_i > x)} \to e^{-\theta} \text{ as } x \to \infty.$$

(b) Define b_n such that $\mathbb{P}(X_i > b_n) = 1/n$. Show that

$$\mathbb{P}(b_n(M_n - b_n) \leq x) \to e^{-e^{-x}} \text{ as } n \to \infty,$$

where $M_n = \max_{1\leq m\leq n} X_m$.

(c) Show that $b_n \sim (2\log n)^{1/2}$ and conclude that $M_n/(2\log n)^{1/2} \xrightarrow{\mathbb{P}} 1$.