# 13th Practice class
## Weak convergence of probability distributions

**13.1** Let $S = [0,1]$ and let $\mu_n$ be the discrete measure such that every point $k/n$ has weight $1/(n+1)$ for $k = 0, 1, \ldots, n$. Show that $\mu_n \Rightarrow \mu$, where $\mu$ is the Lebesgue measure on $S$. (Use the definition of the weak convergence, not the equivalent formalisations for distribution functions.)

**Solution** Let $f : S \mapsto \mathbb{R}$ be bounded and continuous. Then $f$ is Riemann integrable on $[0,1]$. Hence,

$$\int_S f(x)\,\mathrm{d}\mu_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \mu_n\left(\left\{\frac{k}{n}\right\}\right) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\frac{1}{n+1}$$
$$= \frac{n}{n+1} \cdot \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} \to \int_0^1 f(x)\,\mathrm{d}x = \int_S f(x)\,\mathrm{d}\mu(x).$$

Since every Riemann integrable function is Lebesgue integrable and the integrals are the same, the last equality holds.

*Remark.* We actually showed that if

$$X_n \sim \mathrm{Uni}\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\} \qquad \text{and} \qquad X \sim \mathrm{Uni}(0,1),$$

then

$$X_n \xrightarrow{d} X.$$

**13.4** Let $c \in \mathbb{R}$ and $X_n$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the following are equivalent.

(i) $X_n \xrightarrow{\mathbb{P}} c$ as $n \to \infty$.

(ii) $X_n \xrightarrow{d} c$ as $n \to \infty$.

**Solution**

(i)⇒(ii) $X_n \xrightarrow{\mathbb{P}} X$ implies $X_n \xrightarrow{d} X$ in general.

(ii)⇒(i) Let $F$ be the distribution function of the constant $c$ random variable. Then

$$F(x) = \begin{cases} 0 & \text{if } x \le c, \\ 1 & \text{if } c < x. \end{cases}$$

Let $\delta > 0$. Since $c - \delta$ and $c + \delta$ are points of continuity of $F$, by weak convergence

$$\mathbb{P}(|X_n - c| > \delta) = \mathbb{P}(X_n < c - \delta) + \mathbb{P}(X_n > c + \delta) \to F(c - \delta) + 1 - F(c + \delta) = 0 + 1 - 1 = 0.$$

**13.5** *(Cramer-Slutsky theorem)* Let $X_n, Y_n, Z_n$, $n \in \mathbb{N}$ and $X$ be random variables over the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{\mathbb{P}} y$, $Z_n \xrightarrow{\mathbb{P}} z$ for some $y, z \in \mathbb{R}$ then $X_nY_n + Z_n \xrightarrow{d} X \cdot y + z$.

**Solution**

(i) We first show that

$$X_n + Z_n \xrightarrow{d} X + z.$$

Let

$$F(x) = \mathbb{P}(X + z < x) \qquad \text{and} \qquad F_n(x) = \mathbb{P}(X_n + Z_n < x)$$

It is enough to show that for any $x \in \mathbb{R}$ and $\varepsilon > 0$

$$F(x - \varepsilon) \le \liminf_{n\to\infty} F_n(x) \le \limsup_{n\to\infty} F_n(x) \le F(x + \varepsilon).$$

For the upper bound, let $0 < \delta_1 \le \varepsilon$ be such that

$$\mathbb{P}(X + z = x + \delta_1) = 0.$$

Since the distribution of $X$ has at most countably many discontinuity points, there is such a $\delta_1$. Then we have the bound

$$\begin{aligned} F_n(x) &= \mathbb{P}(X_n + Z_n < x) \\ &= \mathbb{P}(X_n + Z_n < x, |Z_n - z| > \delta_1) + \mathbb{P}(X_n + Z_n < x, |Z_n - z| \le \delta_1) \\ &\le \underbrace{\mathbb{P}(|Z_n - z| > \delta_1)}_{\to 0} + \underbrace{\mathbb{P}(X_n + z < x + \delta_1)}_{\to F(x+\delta_1)}. \end{aligned}$$

Hence,

$$\limsup_{n\to\infty} F_n(x) \le F(x+\delta_1) \le F(x+\varepsilon).$$

For the lower bound, let $0 < \delta_2 \le \varepsilon$ be such that

$$\mathbb{P}(X + z = x - \delta_2) = 0.$$

Since the distribution of $X$ has at most countably many discontinuity points, there is such a $\delta_2$. Then we have the bound

$$\begin{aligned} 1 - F_n(x) &= \mathbb{P}(X_n + Z_n \ge x) \\ &= \mathbb{P}(X_n + Z_n \ge x, |Z_n - z| > \delta_2) + \mathbb{P}(X_n + Z_n \ge x, |Z_n - z| \le \delta_2) \\ &\le \mathbb{P}(|Z_n - z| > \delta_1) + \mathbb{P}(X_n + z \ge x - \delta_2) \\ &= \mathbb{P}(|Z_n - z| > \delta_1) + 1 - \mathbb{P}(X_n + z < x - \delta_2). \end{aligned}$$

After rearranging

$$\underbrace{\mathbb{P}(X_n + z < x - \delta_2)}_{\to F(x-\delta_2)} - \underbrace{\mathbb{P}(|Z_n - z| > \delta_1)}_{\to 0} \le F_n(x).$$

Hence,

$$F(x-\varepsilon) \le F(x-\delta_2) \le \liminf_{n\to\infty} F_n(x).$$

(ii) We now show that

$$X_n \cdot Y_n \xrightarrow{d} y \cdot X.$$

Notice

$$X_n \cdot Y_n = y \cdot X_n + X_n(Y_n - y).$$

Hence, by part (i) it is enough to show that

$$y \cdot X_n \xrightarrow{d} y \cdot X \qquad \text{and} \qquad X_n(Y_n - y) \xrightarrow{\mathbb{P}} 0.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Then $x \mapsto f(y \cdot x)$ is also bounded and continuous. Thus,

$$\mathbb{E}(f(y \cdot X_n)) \to \mathbb{E}(f(y \cdot X)).$$

This exactly means

$$y \cdot X_n \xrightarrow{d} y \cdot X.$$

We know that $(X_n)_{n=1}^{\infty}$ is tight. Thus, for any $\varepsilon > 0$ there exits $0 < M < \infty$ such that for all $n \ge 1$

$$\mathbb{P}(|X_n| > M) < \varepsilon.$$

Then for any $\delta > 0$

$$\begin{aligned} \mathbb{P}(|X_n(Y_n - y)| > \delta) &= \mathbb{P}(|X_n(Y_n - y)| > \delta, |X_n| > M) + \mathbb{P}(|X_n(Y_n - y)| > \delta, |X_n| \le M) \\ &\le \mathbb{P}(|X_n| > M) + \mathbb{P}\left(|Y_n - y| > \frac{\delta}{M}\right) \\ &\le \varepsilon + \underbrace{\mathbb{P}\left(|Y_n - y| > \frac{\delta}{M}\right)}_{\to 0}. \end{aligned}$$

Hence, for any $\varepsilon > 0$

$$\limsup_{n\to\infty} \mathbb{P}(|X_n(Y_n - y)| > \delta) \le \varepsilon.$$

It follows

$$X_n(Y_n - y) \xrightarrow{\mathbb{P}} 0.$$

Combining (i) and (ii) we get $X_n \cdot Y_n + Z_n \xrightarrow{d} X_n \cdot y + z$.

**13.7** *(Limit theorems for maximum of i.i.d random variables)* Let $X_1, X_2, \ldots$ be i.i.d random variables with distribution function $F(x) := \mathbb{P}(X_i < x)$. Let $M_n := \max\{X_1, X_2, \ldots, X_n\}$. The asymptotic behavior of the random variable $M_n$ as $n \to \infty$ depends on then asymptotic of the upper tail of $F(x)$. Show the following limit theorems:

(a) Suppose that $F(x) < 1$ for every $x < \infty$ and $\lim_{x\to\infty} x^\alpha(1 - F(x)) = b$ for some $\alpha, b \in (0, \infty)$ Show that the distribution of $n^{-1/\alpha}M_n$ converges weakly to:

$$\mathbb{P}\Big(n^{-1/\alpha}M_n < x\Big) \to \mathbb{1}_{\{x>0\}} \exp\left(-bx^{-\alpha}\right).$$

(b) Suppose that $F(x_0) = 1$ and $F(x) < 1$ for every $x < x_0$, moreover, $\lim_{x\to x_0}(x_0-x)^{-\alpha}(1-F(x)) = b$ for some $\alpha, b \in (0, \infty)$ Then the distribution of $n^{1/\alpha}(x_0 - M_n)$ converges weakly to:

$$\mathbb{P}\Big(n^{1/\alpha}(x_0 - M_n) < x\Big) \to \mathbb{1}_{\{x>0\}} \left(1 - \exp\left(-bx^\alpha\right)\right).$$

(c) Suppose that $F(x) < 1$ for every $x < \infty$ and $\lim_{x\to\infty} e^{\lambda x}(1 - F(x)) = b$ for some $\lambda, b \in (0, \infty)$ Then the distribution of $M_n - \lambda^{-1}\log n$ converges weakly to:

$$\mathbb{P}\Big(M_n - \lambda^{-1}\log n < x\Big) \to \exp\Big(-be^{-\lambda x}\Big).$$

**Solution**

(a) Notice that $x \mapsto \exp(-bx^{-\alpha}) \cdot \mathbb{1}\{x > 0\}$ is continuous at every $x \in \mathbb{R}$. Since the variables are i.i.d.,

$$\begin{aligned}
\mathbb{P}(M_n < x) &= \mathbb{P}(\max\{X_1, \ldots, X_n\} < x)\\
&= \mathbb{P}(X_1 < x, \ldots, X_n < x)\\
&= \mathbb{P}(X_1 < x) \cdots \mathbb{P}(X_n < x)\\
&= F(x)^n.
\end{aligned}$$

Hence, for any $x \in \mathbb{R}$

$$\mathbb{P}(n^{-\frac{1}{\alpha}}M_n < x) = \mathbb{P}(M_n < xn^{\frac{1}{\alpha}}) = F(xn^{\frac{1}{\alpha}})^n.$$

If $x \le 0$, then since $0 \le F(0) < 1$,

$$0 \le F(xn^{\frac{1}{\alpha}})^n \le F(0)^n \to 0.$$

If $x > 0$, then

$$F(xn^{\frac{1}{\alpha}})^n = (1 - (1 - F(xn^{\frac{1}{\alpha}}))^n \to \exp(-bx^{-\alpha}),$$

since

$$\lim_{n\to\infty} n(1 - F(xn^{\frac{1}{\alpha}})) = \lim_{n\to\infty} x^{-\alpha} \underbrace{\left(xn^{\frac{1}{\alpha}}\right)^\alpha (1 - F(xn^{\frac{1}{\alpha}}))}_{\to b,\text{ since } xn^{1/\alpha} \to \infty \text{ as } n \to \infty} = bx^{-\alpha}.$$

Therefore, for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}(n^{-\frac{1}{\alpha}}M_n < x) = \exp(-bx^{-\alpha}) \cdot \mathbb{1}\{x > 0\}.$$