14th Practice Class

Method of characteristic functions and the Central Limit Theorem

14.4 Let $X_1, X_2, \dots$ be non-negative i.i.d random variables with $\mathbb{E}(X_i) = 1$ and $\mathbb{D}^2(X_i) = \sigma^2$. Show that

$$2(\sqrt{S_n} - \sqrt{n}) \xrightarrow{d} N(0, \sigma^2).$$

Solution Notice that

$$(\sqrt{S_n} - \sqrt{n}) \cdot (\sqrt{S_n} + \sqrt{n}) = S_n - n.$$

Hence, by the Cramér-Slutsky theorem

$$2(\sqrt{S_n} - \sqrt{n}) = 2\frac{S_n - n}{\sqrt{S_n} + \sqrt{n}} = \underbrace{\frac{2}{\sqrt{\frac{S_n}{n}} + 1}}_{\xrightarrow{\text{a.s.}} 1 \text{ by the SLLN}} \cdot \underbrace{\frac{S_n - n}{\sqrt{n}}}_{\xrightarrow{d} \mathcal{N}(0,\sigma^2) \text{ by the CLT}} \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

14.8 Show with the method of characteristic functions that $\text{Bin}(n, p_n) \Rightarrow \text{Poi}(\lambda)$ as $n \to \infty$ if $\lim_{n\to\infty} np_n = \lambda$.

Solution The characteristic function of $\text{Bin}(n, p_n)$ is

$$\phi_n(u) = (p_n e^{iu} + 1 - p_n)^n = (1 + p_n(e^{iu} - 1))^n.$$

The characteristic function of $\text{Poi}(\lambda)$ is

$$\phi(u) = e^{\lambda(e^{iu}-1)}.$$

Since

$$\lim_{n\to\infty} n \cdot p_n(e^{iu} - 1) = \lambda(e^{iu} - 1),$$

we have

$$\lim_{n\to\infty} \phi_n(u) = \lim_{n\to\infty} (1 + p_n(e^{iu} - 1))^n = e^{\lambda(e^{iu}-1)} = \phi(u).$$

This exactly means that

$$\text{Bin}(n, p_n) \Rightarrow \text{Poi}(\lambda).$$

14.15 Show that

$$e^{-n}\left(1 + n + \frac{n^2}{2!} + \cdots \frac{n^n}{n!}\right) \to \frac{1}{2} \text{ as } n \to \infty.$$

Solution Notice that if $X_n \sim \text{Poi}(n)$, then

$$e^{-n}\left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) = \sum_{k=0}^{n} e^{-n}\frac{n^k}{k!} = \mathbb{P}(X_n \le n).$$

Notice that if $Y_1, Y_2, \dots$ are i.i.d. with distribution Poi(1), then

$$X_n \stackrel{d}{=} Y_1 + \cdots + Y_n.$$

Since $\mathbb{E}(Y_1) = \mathbb{D}^2(Y_1) = 1$, by the CLT

$$\frac{Y_1 + \cdots + Y_n - n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Since the cumulative distribution function Φ of $\mathcal{N}(0, 1)$ is continuous at zero,

$$\mathbb{P}(X_n \le n) = \mathbb{P}(Y_1 + \cdots + Y_n \le n) = \mathbb{P}\left(\frac{Y_1 + \cdots + Y_n - n}{\sqrt{n}} \le 0\right) \to \Phi(0) = \frac{1}{2}.$$