1st Midterm retake

Working time: 45 minutes. Notes and electronic devices are not allowed.

Ex1. *(12 points)* May B. Dunn is a student in mathematics on BUTE. She tries to pass the Probability Theory 2 course. First, she needs to get the signature in the practical part. If she fails in one semester, she tries again in the next one. The semesters are independent and in each of the semesters the probability that she gets the signature is 1/3. If she gets the signature she will try the oral exam on the theory. Again, if she fails she tries in the next semester, the semesters are independent and in each of the semesters, the probability that she passes the oral exam is 1/4. Find the distribution of the number of semesters required for May B. Dunn to pass.

Solution Let X denote the number of semesters required to get the signature and Y the number of semesters required to pass the exam after getting the signature. Then $\mathbb{P}(X=k)=\frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$ $(k=1,2,\dots)$ and $\mathbb{P}(Y=l)=\frac{1}{4}\left(\frac{3}{4}\right)^{l}$ $(l=0,1,\dots)$. The the total number of semesters required for May B. Dunn to pass is $Z:=X+Y$. Using that X and Y are independent for any $n=1,2,\dots$

$$\begin{aligned}
\mathbb{P}(Z=n)&=\sum_{k=1}^{n}\mathbb{P}(X=k)\,\mathbb{P}(Y=n-k)\\
&=\sum_{k=1}^{n}\frac{1}{3}\left(\frac{2}{3}\right)^{k-1}\frac{1}{4}\left(\frac{3}{4}\right)^{n-k}\\
&=\frac{1}{12}\left(\frac{3}{4}\right)^{n}\sum_{k=1}^{n}\left(\frac{2}{3}\right)^{k-1}\left(\frac{4}{3}\right)^{k}\\
&=\frac{1}{8}\left(\frac{3}{4}\right)^{n}\sum_{k=1}^{n}\left(\frac{8}{9}\right)^{k}\\
&=\frac{1}{8}\left(\frac{3}{4}\right)^{n}\frac{8}{9}\frac{1-\left(\frac{8}{9}\right)^{n}}{1-\frac{8}{9}}\\
&=\left(\frac{3}{4}\right)^{n}-\left(\frac{3}{4}\right)^{n}\left(\frac{8}{9}\right)^{n}=\left(\frac{3}{4}\right)^{n}-\left(\frac{2}{3}\right)^{n}.
\end{aligned}$$

Ex2. *(18 points)* Let $X_1,X_2,\dots$ be a sequence of independent random variables with distribution $X_k\sim\mathrm{Uni}(0,1/k)$. Denote $S_n=X_1+\dots+X_n$. Show that

$$\frac{S_n}{\log(n)}\xrightarrow{\mathbb{P}}\frac{1}{2}\quad\text{as } n\to\infty.$$

(Hint: $\sum_{k=1}^{n}k^{-1}-\log(n)\to\gamma\in(0,1)$ as $n\to\infty$.)

Solution 1 The expectation and the variance of X_k are

$$\mathbb{E}(X_k)=\frac{1}{2k}\qquad\text{and}\qquad\mathbb{D}^2(X_k)=\frac{1}{12k^2}.$$

Hence

$$\mathbb{E}(S_n)=\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}\qquad\text{and}\qquad\mathbb{D}^2(S_n)=\frac{1}{12}\sum_{k=1}^{n}\frac{1}{k^2}.$$

For any $\delta>0$, since $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}=\infty$ and $\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$,

$$\mathbb{P}\left(\left|\frac{S_n}{\sum_{k=1}^{n}\frac{1}{k}}-\frac{1}{2}\right|\geq\delta\right)=\underbrace{\mathbb{P}\left(|S_n-\mathbb{E}(S_n)|\geq\delta\sum_{k=1}^{n}\frac{1}{k}\right)\leq\frac{\frac{1}{12}\sum_{k=1}^{n}\frac{1}{k^2}}{\delta^2\left(\sum_{k=1}^{n}\frac{1}{k}\right)^2}}_{\text{Chebyshev's inequality}}\to 0\quad\text{as } n\to\infty.$$

Thus

$$\frac{S_n}{\sum_{k=1}^n \frac{1}{k}} \xrightarrow{\mathbb{P}} \frac{1}{2} \quad \text{as } n \to \infty.$$

Moreover, we know that

$$\sum_{k=1}^n \frac{1}{k} \sim \log(n) \quad \text{as } n \to \infty.$$

Thus

$$\frac{\sum_{k=1}^n \frac{1}{k}}{\log(n)} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty.$$

We conclude

$$\frac{S_n}{\log(n)} = \underbrace{\frac{S_n}{\sum_{k=1}^n \frac{1}{k}}}_{\xrightarrow{\mathbb{P}} \frac{1}{2}} \cdot \underbrace{\frac{\sum_{k=1}^n}{\log(n)}}_{\xrightarrow{\mathbb{P}} 1} \xrightarrow{\mathbb{P}} \frac{1}{2} \quad \text{as } n \to \infty.$$

Solution 2 The expectation and the variance of X_k are

$$\mathbb{E}(X_k) = \frac{1}{2k} \qquad \text{and} \qquad \mathbb{D}^2(X_k) = \frac{1}{12k^2}.$$

Hence

$$\mathbb{E}(S_n) = \frac{1}{2}\sum_{k=1}^n \frac{1}{k} \qquad \text{and} \qquad \mathbb{D}^2(S_n) = \frac{1}{12}\sum_{k=1}^n \frac{1}{k^2}.$$

For any $\delta > 0$,

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{S_n}{\log(n)} - \frac{1}{2}\right| \geq \delta\right) &= \mathbb{P}\left(\left|S_n - \frac{1}{2}\log(n)\right| \geq \delta \log(n)\right) \\
&= \mathbb{P}\left(\left|S_n - \frac{1}{2}\sum_{k=1}^n\right| \geq \delta \log(n) - \frac{1}{2}\sum_{k=1}^n \frac{1}{k} + \frac{1}{2}\log(n)\right) \\
&= \mathbb{P}\left(|S_n - \mathbb{E}(S_n)| \geq \delta \log(n) - \frac{1}{2}\sum_{k=1}^n \frac{1}{k} + \frac{1}{2}\log(n)\right).
\end{aligned}$$

Therefore, using Chebyshev's inequality

$$\mathbb{P}\left(\left|\frac{S_n}{\log(n)} - \frac{1}{2}\right| \geq \delta\right) \leq \frac{\frac{1}{12}\sum_{k=1}^n \frac{1}{k^2}}{\left(\delta \log(n) - \frac{1}{2}\left(\sum_{k=1}^n \frac{1}{k} - \log(n)\right)\right)^2}.$$

We know that

$$\lim_{n\to\infty} \log(n) = \infty, \qquad \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6}, \qquad \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{k} - \log(n) = \gamma \in (0,1).$$

Hence

$$\lim_{n\to\infty} \frac{\frac{1}{12}\sum_{k=1}^n \frac{1}{k^2}}{\left(\delta \log(n) - \frac{1}{2}\left(\sum_{k=1}^n \frac{1}{k} - \log(n)\right)\right)^2} = 0.$$

We conclude

$$\frac{S_n}{\log(n)} \xrightarrow{\mathbb{P}} \frac{1}{2} \quad \text{as } n \to \infty.$$