

1st Midterm

Working time: 45 minutes. Notes and electronic devices are not allowed.

Ex1. Let us consider an equilateral triangle, and let us label the vertices by A , B , and C . A particle on a vertex jumps to the neighbouring vertex in counterclockwise direction with probability p and in clockwise direction with probability $q = 1 - p$. The jumps are independent. Let us suppose that at time zero, there is a particle on vertex A .

- (a) (12 points) Find the probability generating function of the first return time to vertex A .
 (b) (6 points) What is the expectation of the first return time?

Solution

- (a) Let

$$\xi_n = \begin{cases} -1 & \text{if the } n\text{th step is in clockwise direction,} \\ +1 & \text{if the } n\text{th step is in counterclockwise direction.} \end{cases}$$

Moreover, let $X_n \in \{A, B, C\}$ be the position of the particle at time $n \geq 0$. We are looking for the probability generating function of

$$\varrho = \inf\{n \geq 2 : X_n = A\}.$$

The probability generation function by definition

$$\begin{aligned} R(z) &= \mathbb{E}(z^\varrho) \\ &= \underbrace{\mathbb{E}(z^\varrho \mid \xi_1 = +1)}_{(1)} p + \underbrace{\mathbb{E}(z^\varrho \mid \xi_1 = -1)}_{(2)} q, \end{aligned}$$

where

$$\begin{aligned} \mathbb{E}(z^\varrho \mid \xi_1 = +1) &= \mathbb{E}(z^\varrho \mid \xi_1 = +1, \xi_2 = +1)p + \mathbb{E}(z^\varrho \mid \xi_1 = +1, \xi_2 = -1)q \\ &= pz \mathbb{E}(z^\varrho \mid \xi_1 = -1) + qz^2, \end{aligned} \quad (1)$$

$$\begin{aligned} \mathbb{E}(z^\varrho \mid \xi_1 = -1) &= \mathbb{E}(z^\varrho \mid \xi_1 = -1, \xi_2 = +1)p + \mathbb{E}(z^\varrho \mid \xi_1 = -1, \xi_2 = -1)q \\ &= pz^2 + qz \mathbb{E}(z^\varrho \mid \xi_1 = +1). \end{aligned} \quad (2)$$

Solving the system of equations (1) and (2)

$$\mathbb{E}(z^\varrho \mid \xi_1 = +1) = \frac{p^2 z^3 + qz^2}{1 - pqz^2} \quad \text{and} \quad \mathbb{E}(z^\varrho \mid \xi_1 = -1) = \frac{q^2 z^3 + pz^2}{1 - pqz^2}.$$

It follows

$$R(z) = \frac{p^2 z^3 + qz^2}{1 - pqz^2} p + \frac{q^2 z^3 + pz^2}{1 - pqz^2} q = \frac{(p^3 + q^3)z^3 + 2pqz^2}{1 - pqz^2}.$$

- (b) We know that

$$\begin{aligned} \mathbb{E}(\varrho) &= R'(1) \\ &= \left. \frac{d}{dz} \frac{(p^3 + q^3)z^3 + 2pqz^2}{1 - pqz^2} \right|_{z=1} \\ &= \left. \frac{(1 - pqz^2)(3(p^3 + q^3)z^2 + 4pqz) + 2pqz((p^3 + q^3)z^3 + 2pqz^2)}{(1 - pqz^2)^2} \right|_{z=1} \\ &= \frac{3(1 - pq)^2}{(1 - pq)^2} \\ &= 3. \end{aligned}$$

Ex2. (12 points) Let X_1, \dots, X_9 be independent random variables with distribution $\text{Uni}(0, 1)$. Moreover, let $Y = \sqrt[9]{X_1 \cdots X_9}$. Using Chebysev's inequality, give a lower estimate for the probability

$$\mathbb{P}(e^{-5/3} < Y < e^{-1/3})!$$

Solution Notice that

$$\mathbb{P}(e^{-5/3} < Y < e^{-1/3}) = \mathbb{P}\left(-\frac{5}{3} < \log(Y) < -\frac{1}{3}\right) = \mathbb{P}\left(-\frac{5}{3} < \frac{\log(X_1) + \cdots + \log(X_9)}{9} < -\frac{1}{3}\right).$$

The first moment is

$$\mathbb{E}(\log(X_j)) = \int_0^1 \log(x) \, dx = -1$$

and the second moment is

$$\mathbb{E}(\log(X_j)^2) = \int_0^1 \log(x)^2 \, dx = 2.$$

Therefore,

$$\mathbb{E}(\log(X_j)) = -1 \quad \text{and} \quad \mathbb{D}^2(\log(X_j)) = 2 - (-1)^2 = 1.$$

Using the independence

$$\mathbb{E}(\log(Y)) = \frac{1}{9} \cdot 9 \cdot \mathbb{E}(\log(X_j)) = -1 \quad \text{and} \quad \mathbb{D}^2(\log(Y)) = \frac{1}{9^2} \cdot 9 \cdot \mathbb{D}^2(\log(X_j)) = \frac{1}{9}.$$

Then

$$\mathbb{P}\left(-\frac{5}{3} < \log(Y) < -\frac{1}{3}\right) = \mathbb{P}\left(-\frac{2}{3} < \log(Y) + 1 < \frac{2}{3}\right) = \mathbb{P}\left(|\log(Y) - \mathbb{E}(\log(Y))| < \frac{2}{3}\right).$$

Therefore, using Chebysev's inequality

$$\begin{aligned} \mathbb{P}\left(|\log(Y) - \mathbb{E}(\log(Y))| < \frac{2}{3}\right) &= 1 - \mathbb{P}\left(|\log(Y) - \mathbb{E}(\log(Y))| \geq \frac{2}{3}\right) \\ &\geq 1 - \left(\frac{3}{2}\right)^2 \underbrace{\mathbb{D}^2(\log(Y))}_{=1/9} \\ &= \frac{3}{4}. \end{aligned}$$