2nd Midterm retake

Working time: 45 minutes. Notes and electronic devices are not allowed.

Ex1. *(12 points)* We make infinitely many independent experiments. The probability that the nth experiment is successful is $n^{-\alpha}$, where $0 < \alpha < 1$. Let $k \geq 1$. It makes us happy if it happens infinitely often that we have k consecutive successful experiments. What is the probability that we are happy (depending on k and α)?

Solution Let for $n \geq 1$

$$X_n = \mathbb{1}\left\{\text{the } n\text{th experiment is successful}\right\}.$$

(i) If $k\alpha > 1$, then for $n \geq 1$ let

$$A_n = \{X_n = X_{n+1} = \dots X_{n+k-1} = 1\}.$$

It follows

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \prod_{j=0}^{k-1} (n+j)^{-\alpha} \leq \sum_{n=1}^{\infty} n^{-k\alpha} < \infty.$$

Hence, by the first Borel-Cantelli lemma

$$\mathbb{P}(\text{we are happy}) = \mathbb{P}(A_n \text{ happens for infinitely many } n) = 0.$$

(ii) If $k\alpha \leq 1$, then for $n \geq 0$ let

$$B_n = \Big\{X_{nk+1} = X_{nk+1} = \dots = X_{(n+1)k} = 1\Big\}.$$

It follows

$$\sum_{n=0}^{\infty} \mathbb{P}(B_n) = \sum_{n=0}^{\infty} \prod_{j=1}^{k} (nk+j)^{-\alpha} \geq \sum_{n=0}^{\infty} (nk+k)^{-\alpha} = k^{-\alpha} \sum_{n=1}^{\infty} n^{-k\alpha} = \infty.$$

Since the events $B_0, B_1, \dots$ are independent, by the second Borel-Cantelli lemma

$$\mathbb{P}(\text{we are happy}) \geq \mathbb{P}(B_n \text{ happens for infinitely many } n) = 1.$$

Ex2. Let X, Y be independent and identically distributed random variables with finite variance. Suppose that

$$\frac{X+Y}{\sqrt{2}} \quad \text{and} \quad \frac{X-Y}{\sqrt{2}}$$

have the same distribution as X (and Y).

(a) *(6 points)* Show that X and Y are symmetric, that is $\mathbb{P}(X < x) = \mathbb{P}(X > -x)$ for every $x \in \mathbb{R}$. What is the expected value of X and Y?

(b) *(12 points)* Show that X and Y are normal.

Solution

(a) Let ϕ denote the characteristic function of X (and Y). Since

$$X \overset{d}{=} \frac{X+Y}{\sqrt{2}} \quad \text{and} \quad X \overset{d}{=} \frac{X-Y}{\sqrt{2}},$$

for every $u \in \mathbb{R}$

$$\phi(u) = \phi\left(\frac{u}{\sqrt{2}}\right)^2 \quad \text{and} \quad \phi(u) = \left|\phi\left(\frac{u}{\sqrt{2}}\right)\right|^2.$$

Hence, for every $u \in \mathbb{R}$

$$\phi\left(\frac{u}{\sqrt{2}}\right)^2 = \left|\phi\left(\frac{u}{\sqrt{2}}\right)^2\right| \in \mathbb{R}_0^+,$$

which can only happen if $\phi(u)$ is real for every $u \in \mathbb{R}$. It follows that X has symmetric distribution.

For the expectation,

$$\mathbb{E}(X) = \frac{1}{\sqrt{2}}(\mathbb{E}(X) + \mathbb{E}(Y)) = \frac{2}{\sqrt{2}}\,\mathbb{E}(X).$$

Hence,

$$\mathbb{E}(X) = 0.$$

(b) We first show by induction that for any $u \in \mathbb{R}$ and $n \geq 1$

$$\phi(u) = \phi\left(\frac{u}{\sqrt{2^n}}\right)^{2^n}.$$

We have seen that it holds for $n = 1$. Suppose it is true for some $n \geq 1$. Then using the induction formula and the first identity in part (a)

$$\phi(u) = \phi\left(\frac{u}{\sqrt{2^n}}\right)^{2^n} = \left(\phi\left(\frac{1}{\sqrt{2}}\frac{u}{\sqrt{2^n}}\right)^2\right)^{2^n} = \phi\left(\frac{u}{\sqrt{2^{n+1}}}\right)^{2^{n+1}}.$$

Therefore, it is true for $n + 1$.

Let $\sigma^2 = \mathbb{E}(X^2) = \mathbb{D}^2(X) < \infty$ (since $\mathbb{E}(X) = 0$). Then as $u \to 0$

$$\phi(u) = 1 - \frac{\sigma^2 u^2}{2} + o(u^2).$$

It follows for every $u \in \mathbb{R}$

$$\phi(u) = \lim_{n\to\infty} \phi\left(\frac{u}{\sqrt{2^n}}\right)^{2^n} = \lim_{n\to\infty}\left(1 - \frac{\sigma^2 u^2}{2 \cdot 2^n} + o\left(\frac{1}{2^n}\right)\right)^{2^n} = \exp\left(-\frac{\sigma^2 u^2}{2}\right).$$

This exactly means that

$$X \sim \mathcal{N}(0, \sigma^2).$$