2nd Midterm

Working time: 45 minutes. Notes and electronic devices are not allowed.

Ex1. Let $X_1, X_2, X_3, \dots$ be independent identically distributed random variables with distribution $\mathrm{Uni}(0, a)$ with some $a \in (0, \infty)$. Consider the series

$$\sum_{n=1}^{\infty} X_1 \cdots X_n.$$

(a) *(12 points)* Show that there exists a unique $a^\star \in (0, \infty)$ such that if $a < a^\star$, then the series converges almost surely and if $a > a^\star$, then the series diverges almost surely! Find the exact value of $a^\star$! (*Hint*: Root test.)

(b) *(6 points)* Does the series converge or diverge when $a = a^\star$? Prove your answer!

Solution

(a) Let us define

$$R = \limsup_{n\to\infty} \sqrt[n]{X_1 \cdots X_n}.$$

Then we know that

$$\{R < 1\} \subseteq \{\text{the series converges}\} \quad \text{and} \quad \{R > 1\} \subseteq \{\text{the series diverges}\}.$$

By the Strong Law of Large Numbers (it can be applied since $\int_0^a \frac{|\log(x)|}{a}\,\mathrm{d}x < \infty$), almost surely

$$\log(R) = \lim_{n\to\infty} \frac{\log(X_1) + \cdots + \log(X_n)}{n} = \mathbb{E}(\log(X_1)) = \int_0^a \frac{\log(x)}{a}\,\mathrm{d}x = \log(a) - 1.$$

Since $x \mapsto e^x$ is continuous,

$$R = \exp\left(\lim_{n\to\infty} \frac{\log(X_1) + \cdots + \log(X_n)}{n}\right) = \frac{a}{e} \quad \text{almost surely.}$$

As a consequence

$$a^\star = e.$$

(b) Let $a = e$. Then

$$\mathbb{E}(\log(X_1)) = \log(e) - 1 = 0 \quad \text{and} \quad \mathbb{E}(\log(X_1)^2) = \int_0^e \frac{\log(x)^2}{e}\,\mathrm{d}x = 1.$$

Let

$$S_n = \log(X_1) + \cdots + \log(X_n).$$

By the Law of the Iterated Logarithm

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log(n)}} = 1 \quad \text{almost surely.}$$

Hence,

$$\limsup_{n\to\infty} S_n = \infty \quad \text{almost surely.}$$

Since $x \mapsto e^x$ is monotone and continuous,

$$\limsup_{n\to\infty} X_1 \cdots X_n = \exp\left(\limsup_{n\to\infty} S_n\right) = \infty \quad \text{almost surely.}$$

The series does not converge since

$$\lim_{n\to\infty} X_1 \cdots X_n \neq 0 \quad \text{almost surely.}$$

Ex2. *(12 points)* Show that the difference of two independent and identically distributed random variables cannot have distribution Uni$(-1, 1)$!

Solution The characteristic function of Uni$(-1, 1)$ is

$$u \mapsto \begin{cases} \dfrac{\sin(u)}{u} & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases}$$

Let X_1, X_2 be independent and identically distributed with characteristic function ϕ. Then the characteristic function of $X_1 - X_2$ is

$$u \mapsto |\phi(u)|^2 .$$

Clearly the characteristic function of $X_1 - X_2$ is a non-negative real valued function, while the characteristic function of Uni$(-1, 1)$ is real valued as well but takes negative values too. For $u = 3\pi/2$

$$\frac{\sin(3\pi/2)}{3\pi/2} = -\frac{2}{3\pi} \neq |\phi(3\pi/2)|^2 \geq 0$$

Therefore, their characteristic functions cannot be the same for all $u \in \mathbb{R}$. Hence, they cannot have the same distribution.