1st Midterm

Working time: 45 minutes

Ex1 (10 points) Thanos killed exactly half of the population with his magic ring in a second. Suppose that there were $2N$ people and exactly N men and N women. The ring choose between people uniformly, that is, it killed every subset of the population with N element with equal probabilities. Let X_N the ratio of men in the survivor population. Show that X_N converges to $1/2$ in probability as $N \to \infty$.

Ex2 (10 points) We put an amoebae into a Petri dish. The amoebas divide in every minute according to the following rule: at the end of every odd minute an amoebae divides into k many amoebas with probability $p_1(k)$, $k \geq 0$ (so it can die), and at the end of every even minute, an amoebae divides with probability $p_2(k)$ to k many amoebas ($k \geq 0$). So after $2n+1$ minutes every amoebas divides independently with prob. $p_1(k)$ into k amoebas and the same happens independently with probability $p_2(k)$ after $2n$ minutes.

(a) Denote $G_1(x)$ and $G_2(x)$ the probability generating functions of the distributions p_1 and p_2 respectively. Find the prob. generating function of the nth generation! (We put the amoebae into the dish in the 0th minute.)

(b) Let p_1 be Geo(1/3) (i.e. $p_1(k) = \frac{1}{3}\left(\frac{2}{3}\right)^k$ for $k \geq 0$), and let p_2 be Geo(3/4) ($p_2(k) = \frac{3}{4}\left(\frac{1}{4}\right)^k$ for $k \geq 0$). Find the probability that the amoebas eventually die out.

Ex3 (10 points) Let X be the uniform random variable on the set $\{0, 1, \ldots, n-1\}$. Show that if n is not a prime then there exist independent $\mathbb{N}$-valued random variables Y, Z such that $X = Y + Z$. (Hint: Decompose with respect to the remaining classes.)