1st Midterm

Working time: 45 minutes

Ex1 (10 points) Thanos killed exactly half of the population with his magic ring in a second. Suppose that there were $2N$ people and exactly N men and N women. The ring choose between people uniformly, that is, it killed every subset of the population with N element with equal probabilities. Let X_N the ratio of men in the survivor population. Show that X_N converges to $1/2$ in probability as $N \to \infty$.

Solution

Let use denote for $k = 1, 2, \dots, N$

$$\xi_k = \mathbb{1}\left[\text{the } k\text{th man survived}\right].$$

Then the total men who survived is

$$S_N = \sum_{k=1}^{N} \xi_k.$$

Moreover

$$\mathbb{P}(\xi_k = 1) = \mathbb{P}(\text{the } k\text{th man survived}) = \frac{\binom{2N-1}{N-1}}{\binom{2N}{N}} = \frac{1}{2}.$$

Thus

$$\mathbb{E}(\xi_k) = \frac{1}{2} \qquad \text{and} \qquad \mathbb{D}^2(\xi_k) = \frac{1}{4}.$$

Moreover for any $k \neq l$

$$\mathbb{E}(\xi_k \xi_l) = \mathbb{P}(\text{the } k\text{th and } l\text{th man survived}) = \frac{\binom{2N-2}{N-2}}{\binom{2N}{N}} = \frac{N-1}{4N-2} < \frac{1}{4}.$$

Thus

$$\mathbf{Cov}(\xi_k, \xi_l) = \mathbb{E}(\xi_k \xi_l) - \mathbb{E}(\xi_k)\mathbb{E}(\xi_l) < \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{2} < 0.$$

Then

$$\mathbb{E}(S_N) = \frac{N}{2} \qquad \text{and} \qquad \mathbb{D}^2(S_N) = \frac{N}{4} + \underbrace{2 \sum_{k=2}^{N} \sum_{l=1}^{k-1} \mathbf{Cov}(\xi_k, \xi_l)}_{<0} < \frac{N}{4}.$$

Then for any $\delta > 0$ using Chebyshev's inequality

$$\mathbb{P}\left(\left|X_N - \frac{1}{2}\right| > \delta\right) = \mathbb{P}\left(\left|S_N - \frac{N}{2}\right| > \delta N\right) = \mathbb{P}(|S_N - \mathbb{E}(S_N)| > \delta N)$$
$$\leq \frac{\mathbb{D}^2(S_N)}{\delta^2 N^2} < \frac{\frac{N}{4}}{\delta^2 N^2} = \frac{1}{4\delta^2} \cdot \frac{1}{N} \to 0 \quad \text{as } N \to \infty.$$

Which exactly means

$$X_N \xrightarrow{\mathbb{P}} \frac{1}{2} \quad \text{as } N \to \infty.$$

Ex2 (10 points) We put an amoebae into a Petri dish. The amoebas divide in every minute according to the following rule: at the end of every odd minute an amoebae divides into k many amoebas with probability $p_1(k)$, $k \geq 0$ (so it can die), and at the end of every even minute, an amoebae divides with probability $p_2(k)$ to k many amoebas ($k \geq 0$). So after $2n+1$ minutes every amoebas divides independently with prob. $p_1(k)$ into k amoebas and the same happens independently with probability $p_2(k)$ after $2n$ minutes.

(a) Denote $G_1(x)$ and $G_2(x)$ the probability generating functions of the distributions p_1 and p_2 respectively. Find the prob. generating function of the nth generation! (We put the amoebae into the dish in the 0th minute.)

(b) Let p_1 be Geo(1/3) (i.e. $p_1(k) = \frac{1}{3}\left(\frac{2}{3}\right)^k$ for $k \geq 0$), and let p_2 be Geo(3/4) ($p_2(k) = \frac{3}{4}\left(\frac{1}{4}\right)^k$ for $k \geq 0$). Find the probability that the amoebas eventually die out.

Solution

(a) Let $\xi^{(1)}_{n,k}$ be i.i.d. with distribution p_1 and $\xi^{(2)}_{n,k}$ be i.i.d. with distribution p_2. Let Z_n denote the size of the nth generation. Then $Z_0 = 1$ by definition. Let $P_n(x)$ denote the PGF of Z_n. Then clearly

$$P_0(x) = x.$$

Now consider two cases

(i) If n is odd

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi^{(1)}_{n-1,k},$$

thus

$$P_n(x) = P_{n-1}\left(G_1(x)\right).$$

(ii) If n is even

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi^{(2)}_{n-1,k},$$

thus

$$P_n(x) = P_{n-1}\left(G_2(x)\right).$$

Then clearly we have

$$P_n(x) = \begin{cases} \underbrace{(G_2 \circ G_1) \circ \cdots \circ (G_2 \circ G_1)}_{k}(x) & \text{if } n = 2k \\ G_1 \circ \underbrace{(G_2 \circ G_1) \circ \cdots \circ (G_2 \circ G_1)}_{k}(x) & \text{if } n = 2k+1 \end{cases}$$

(b) In this case we know that

$$G_1(x) = \frac{\frac{1}{3}}{1 - \frac{2}{3}s} = \frac{1}{3-2s}$$

and

$$G_2(x) = \frac{\frac{3}{4}}{1 - \frac{1}{4}s} = \frac{3}{4-s}.$$

Then

$$G_2 \circ G_1(x) = \frac{3}{4 - 2\frac{1}{3-2x}} = \frac{9-6x}{11-8x}.$$

Moreover we know that

$$\mathbb{P}(\text{the amoebas eventually die out}) = \lim_{n\to\infty} P_n(0).$$

First let us calculate the limit

$$\lim_{n\to\infty} P_{2n}(0).$$

This is exactly the smallest fixed point of $G_2 \circ G_1(x)$. Solving the equation

$$\frac{9-6x}{11-8x} = x$$

the solutions are

$$x_1 = 1 \qquad \text{and} \qquad x_2 = \frac{9}{8}.$$

Thus

$$\lim_{n\to\infty} P_{2n}(0) = 1.$$

Moreover

$$\lim_{n\to\infty} P_{2n+1}(0) = \lim_{n\to\infty} G_1\left(P_{2n}(0)\right) = G_1\left(\lim_{n\to\infty} P_{2n}(0)\right) = G_1(1) = 1.$$

Thus

$$\lim_{n\to\infty} P_n(0) = 1,$$

which exactly means

$$\mathbb{P}(\text{the amoebas eventually die out}) = 1.$$

Ex3 (10 points) Let X be the uniform random variable on the set $\{0, 1, \ldots, n-1\}$. Show that if n is not a prime then there exist independent $\mathbb{N}$-valued random variables Y, Z such that $X = Y + Z$. (Hint: Decompose with respect to the remaining classes.)

Solution

Since n is not a prime number there exist two positive integers $a, b \geq 2$ such that

$$n = ab.$$

Let us define

$$Y = \left\lfloor \frac{X}{a} \right\rfloor \qquad \text{and} \qquad Z = X \mod a.$$

Then clearly

$$Y \in \{0, 1, \ldots, b-1\} \qquad \text{and} \qquad Z \in \{0, 1, \ldots, a-1\}$$

plus just by the definition of Y and Z

$$X = Y + Z.$$

Now we need to show that Y and Z are independent. Notice that for $k \in \{0, 1, \ldots, b-1\}$

$$\mathbb{P}(Y = k) = \mathbb{P}\left(\left\lfloor \frac{X}{a} \right\rfloor = k\right) = \mathbb{P}(ak \leq X < a(k+1))$$

$$= \sum_{m=ak}^{a(k+1)-1} \mathbb{P}(X = m) = \frac{a(k+1) - 1 - ak + 1}{n} = \frac{a}{n} = \frac{1}{b}.$$

Moreover for any $l \in \{0, 1, \ldots, a-1\}$

$$\mathbb{P}(Z = l) = \mathbb{P}(X \mod a = l) = \sum_{m=0}^{b-1} \mathbb{P}(X = ma + l) = \frac{b - 1 + 1}{n} = \frac{b}{n} = \frac{1}{a}.$$

Then for any $k \in \{0, 1, \ldots, b-1\}$ and $l \in \{0, 1, \ldots, a-1\}$

$$\mathbb{P}(Y = k, Z = l) = \mathbb{P}(X = ak + l) = \frac{1}{n} = \frac{1}{a} \cdot \frac{1}{b} = \mathbb{P}(Y = k) \cdot \mathbb{P}(Z = l).$$

Thus Y and Z are independent. In addition we also proved that

$$Y \sim \text{UNI}\{0, 1, \ldots, b-1\} \qquad \text{and} \qquad Z \sim \text{UNI}\{0, 1, \ldots, a-1\}.$$