2nd Midterm

Working time: 45 minutes

Ex1 Let $X_2, X_3, \ldots$ be independent random variables such that X_n has distribution $\mathrm{Exp}(\log(n))$.

(a) (7 points) Show that

$$X_n \xrightarrow{\mathbb{P}} 0 \qquad \text{and} \qquad X_n \xrightarrow{L^2} 0 \qquad \text{as } n \to \infty.$$

(b) (13 points) Show that

$$\mathbb{P}(\limsup_{n\to\infty} X_n = 1) = 1.$$

(Hint: For which values of $c > 0$ do the events $\{X_n > c\}$ happen infinitely often?)

Solution

(a) Since $X_n \sim \mathrm{Exp}(\log(n))$, we know that

$$\mathbb{E}(X_n) = \frac{1}{\log(n)} \qquad \text{and} \qquad \mathbb{D}^2(X_n) = \frac{1}{\log(n)^2}.$$

Hence,

$$\mathbb{E}(X_n^2) = \mathbb{D}^2(X_n) + \mathbb{E}(X_n)^2 = \frac{1}{\log(n)^2} + \frac{1}{\log(n)^2} = \frac{2}{\log(n)^2} \to 0 \quad \text{as } n \to \infty.$$

It exactly means that

$$X_n \xrightarrow{L^2} 0 \quad \text{as } n \to \infty.$$

Moreover, this implies $X_n \xrightarrow{L^1} 0$, which implies $X_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$.

(b)
- Let $c > 1$. Then

  $$\sum_{n=1}^{\infty} \mathbb{P}(X_n > c) = \sum_{n=1}^{\infty} e^{-\log(n)\cdot c} = \sum_{n=1}^{\infty} \frac{1}{n^c} < \infty.$$

  Therefore, by the first Borel-Cantelli lemma

  $$\mathbb{P}(X_n > c \text{ for infinitely many } n) = 0.$$

  It follows

  $$\mathbb{P}(\limsup_{n\to\infty} X_n \le 1) = 1.$$

- Let $0 < c < 1$. Then

  $$\sum_{n=1}^{\infty} \mathbb{P}(X_n > c) = \sum_{n=1}^{\infty} e^{-\log(n)\cdot c} = \sum_{n=1}^{\infty} \frac{1}{n^c} = \infty.$$

  Since the events $\{X_n > c\}$ are independent, by the second Borel-Cantelli lemma

  $$\mathbb{P}(X_n > c \text{ for infinitely many } n) = 1.$$

  It follows

  $$\mathbb{P}(\limsup_{n\to\infty} X_n \ge 1) = 1.$$

We conclude

$$\mathbb{P}(\limsup_{n\to\infty} X_n = 1) = 1.$$

Ex2 (10 points) Let X_1, X_2 be i.i.d. random variables with distribution $\mathrm{Uni}(0,1)$, and let $Y = \max\{X_1, X_2\}$. Find the characteristic function of Y.

Solution Clearly, $\mathbb{P}(0 \le Y \le 1) = 1$. Let $x \in [0,1]$. Using that $\max\{X_1, X_2\} < x$ if both variables are strictly less than x and the independence, we get

$$\begin{aligned} F(x) &= \mathbb{P}(Y < x) \\ &= \mathbb{P}(\max\{X_1, X_2\} < x) \\ &= \mathbb{P}(X_1 < x, X_2 < x) \\ &= \mathbb{P}(X_1 < x) \cdot \mathbb{P}(X_2 < x) \\ &= x \cdot x \\ &= x^2. \end{aligned}$$

Hence, the probability density function of Y at $x \in [0,1]$

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) = 2x.$$

Therefore, the density of Y is

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic function of Y at any real $u \neq 0$ is

$$\begin{aligned} \phi(u) &= \mathbb{E}(e^{iuY}) \\ &= \int_0^1 e^{iuy} \cdot 2y \, \mathrm{d}y \\ &= \left[\frac{e^{iuy}}{iu} \cdot 2y \right]_{y=0}^{y=1} - \int_0^1 \frac{2e^{iuy}}{iu} \, \mathrm{d}y \\ &= \frac{2e^{iu}}{iu} + \left[\frac{2e^{iuy}}{u^2} \right]_{y=0}^{y=1} \\ &= \frac{2e^{iu}}{iu} + \frac{2e^{iu}}{u^2} - \frac{2}{u^2} \\ &= \frac{2ue^{iu} + 2i(e^{iu} - 1)}{iu^2} \\ &= \frac{2(e^{iu} - 1) - 2iue^{iu}}{u^2} \end{aligned}$$

Moreover, by definition $\phi(0) = 1$. We conclude

$$\phi(u) = \begin{cases} \dfrac{2(e^{iu} - 1) - 2iue^{iu}}{u^2} & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases}$$