Convergence of types thm: $\frac{}{a_n} \to A_k$; $\frac{b_{nk} - b_n}{a_n} \to B_k$ and $H \overset{d}{=} \frac{\max(H_1, \dots, H_k) - B_k}{A_k}$

<u>Thm</u> (Fisher-Tippett 1928, Gnedenko 1943)

Let $X_1, X_2, \dots$ be an iid sequence. If there are $a_n > 0$ and $b_n \in \mathbb{R}$ s.t.

$$\frac{M_n - b_n}{a_n} \overset{d}{\Rightarrow} H \quad (n \to \infty)$$ where H has non-degenerate distribution, then its distribution function is of the type of the following three classes:

- Fréchet : $\Phi_\alpha(x) = \begin{cases} 0 & x \leq 0 \\ \exp(-x^{-\alpha}) & x > 0 \end{cases}$ $\quad \alpha > 0$
- Weibull : $\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha) & x \leq 0 \\ 1 & x > 0 \end{cases}$ $\quad \alpha > 0$
- Gumbel : $\Lambda(x) = \exp(-e^{-x}) \quad x \in \mathbb{R}$

Proof: $G(x) = \mathbb{P}(H < x)$; $F(x) = \mathbb{P}(X_1 < x)$

For any $t > 0$:

$$\frac{M_{[nt]} - b_{[nt]}}{a_{[nt]}} \xrightarrow{d} H \qquad \text{i.e.} \qquad F^{[nt]}(a_{[nt]}x + b_{[nt]}) \longrightarrow G(x)$$ for all points of continuity ~~points~~ of G.

Also $\cancel{\frac{M_{[nt]} - b_n}{}}$ $$F^{[nt]}(a_n x + b_n) = \left(F^n(a_n x + b_n)\right)^{\frac{[nt]}{n}} \longrightarrow G^t(x) \quad \text{—}''\text{—}$$

Convergence of types thm: $\frac{a_n}{a_{[nt]}} \longrightarrow \alpha(t) > 0$ and $\frac{b_n - b_{[nt]}}{a_{[nt]}} \longrightarrow \beta(t) \in \mathbb{R}$ $\forall t > 0$

and $\boxed{G^t(x) = G(\alpha(t)x + \beta(t))}$

Note that $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$ are measurable (limits of measurable functions)

$$G^{ts}(x) = G(\alpha(ts)x + \beta(ts))$$

$$G^{ts}(x) = (G^s(x))^t = (G(\alpha(s)x + \beta(s)))^t = G(\alpha(t)(\alpha(s)x + \beta(s)) + \beta(t))$$
$$= G(\alpha(t)\alpha(s)x + \alpha(t)\beta(s) + \beta(t))$$

$$\Rightarrow \boxed{\alpha(ts) = \alpha(t)\alpha(s)} \quad \text{and} \quad \boxed{\beta(ts) = \alpha(t)\beta(s) + \beta(t) \underset{\uparrow \text{ symmetry}}{=} \alpha(s)\beta(t) + \beta(s) \quad t,s > 0}$$

$f(x) = \log \alpha(e^x)$ i.e. $\alpha(t) = e^{f(\log t)}$

$f(x+y) = f(x) + f(y)$ + measurability $\Rightarrow$ $f(x) = -\theta x$ is the only solution $\theta \in \mathbb{R}$

$\alpha(t) = t^{-\theta}$

Case 1: $\theta = 0$: $\alpha(t) \equiv 1$

$\Rightarrow \beta(ts) = \beta(t) + \beta(s)$ + measurability $\Rightarrow \beta(t) = -c \log t$ $t \in \mathbb{R}$

and $G^t(x) = G(x - c\log t)$

$c \neq 0$ (otherwise G would be degenerate) ; since $t \mapsto G^t(x)$ is nonincreasing $\Rightarrow c > 0$

If there were x_0: $G(x_0) = 1$, then $1 = G(x_0 - c\log t)$ $\forall t \Rightarrow G \equiv 1$ ↯

$\Rightarrow G(x) < 1$ $\forall x$

If there were x_0: $G(x_0) = 0$, then $0 = G(x_0 - c\log t)$ $\forall t \Rightarrow G \equiv 0$ ↯

$\Rightarrow G(x) > 0$ $\forall x$

Take $x = 0$ $\quad G^t(0) = G(-c\log t)$

Set $G(0) = \exp(-e^{-p}) \in (0,1)$ where $p \in \mathbb{R}$ and $u = -c\log t \in (-\infty, \infty)$ i.e. $t = e^{-u/c} \in (0, \infty)$

$$\Rightarrow G(u) = \exp(-e^{-p} \cdot t) = \exp\left(-e^{-(p + \frac{u}{c})}\right) = \Lambda\left(p + \frac{u}{c}\right)$$

Case 2 : $\theta > 0$: $\alpha(t) = t^{-\theta}$ $\alpha(t)\beta(s) + \beta(t) = \alpha(s)\beta(t) + \beta(s)$

If $t \neq 1, s \neq 1$ $\quad \frac{\beta(s)}{1-\alpha(s)} = \frac{\beta(t)}{1-\alpha(t)}$ i.e. $\frac{\beta(t)}{1-\alpha(t)} = c \quad t \neq 1$

$$\beta(t) = c(1 - t^{-\theta})$$

$$G^t(x) = G(\alpha(t)x + \beta(t)) = G(t^{-\theta}x + c(1-t^{-\theta})) = G(t^{-\theta}(x-c)+c)$$

for $x = x + c$: $\quad G^t(x+c) = G(t^{-\theta}x + c)$

Let $H(x) = G(x+c)$, then $\boxed{H^t(x) = H(t^{-\theta}x)}$!

$x = 0$ $\quad t \log H(0) = \log H(0) \quad t > 0 \Rightarrow \log H(0) = \begin{cases} -\infty \\ 0 \end{cases} \quad H(0) = \begin{cases} 0 \\ 1 \end{cases}$

$H(0) = 1$ is impossible (otherwise $\exists x < 0$: $H(x) \in (0,1)$, then $t \mapsto H^t(x) \downarrow$, but $t \mapsto H(t^{-\theta}x)$ ↑)

$\Rightarrow H(0) = 0$

$x = 1$: $H^t(1) = H(t^{-\theta})$. If $H(1) = 0 \Rightarrow H \equiv 0$ ↯ ; if $H(1) = 1 \Rightarrow H \equiv 1$ ↯

$\Rightarrow H(1) \in (0,1)$.

Set $H(1) = \exp(-p^{-1/\theta}) \in (0,1)$ where $p > 0$ and $u = t^{-\theta} \in (0,\infty)$ $\quad t = u^{-1/\theta} \in (0,\infty)$

$$H(t^{-\theta}) = H^t(1)$$

$$H(u) = \exp(-p^{-1/\theta} t) = \exp(-(pu)^{-1/\theta}) = \Phi_{1/\theta}(pu) \qquad \alpha = 1/\theta.$$

Case 3 : $\theta < 0$: similar

Remark

- If $X_1, X_2, \dots$ iid. with Φ_α distr. (Fréchet), then

  $\mathbb{P}(M_n < n^{1/\alpha}x) = (\Phi_\alpha(n^{1/\alpha}x))^n = \exp(-n(n^{1/\alpha}x)^{-\alpha}) = \Phi_\alpha(x)$, that is,

  $M_n \overset{d}{=} n^{1/\alpha} X_1$
- If $X_1, X_2, \dots$ iid. with Ψ_α Weibull distr., then $M_n \overset{d}{=} n^{-1/\alpha} X_1$
- If $X_1, X_2, \dots$ iid. with Λ Gumbel distr., then $M_n \overset{d}{=} X_1 + \ln n$

Ex

Suppose $X > 0$, then the following are equivalent:

1. X has Φ_α Fréchet distribution
2. $\ln X^\alpha$ has Λ Gumbel distribution
3. $-X^{-1}$ has Ψ_α Weibull distribution