Maximum domains of attraction

Examples

① Let $X_1, X_2, \dots$ iid. Cauchy with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$

Then $\bar{F}(x) \sim \frac{1}{\pi x}$ as $x \to \infty$.

$$\mathbb{P}\left(M_n < \frac{nx}{\pi}\right) = \left(1 - \bar{F}\left(\frac{nx}{\pi}\right)\right)^n = \left(1 - \frac{1}{nx} + o\left(\frac{1}{n}\right)\right)^n \to e^{-\frac{1}{x}} = \Phi_1(x)$$

$$\frac{\pi}{n} M_n \overset{d}{\Rightarrow} \Phi_1$$

② Let $X_1, X_2, \dots$ iid. $EXP(1)$. Then $\bar{F}(x) = e^{-x}$.

$$\mathbb{P}(M_n - \ln n < x) = \left(1 - \bar{F}(x + \ln n)\right)^n = \left(1 - \frac{e^{-x}}{n}\right)^n \to e^{-e^{-x}} = \Lambda(x)$$

$$M_n - \ln n \overset{d}{\Rightarrow} \Lambda$$

Def

The distribution function F belongs to the maximum domain of attraction of the extreme value distribution H, if there are $a_n > 0$ and $b_n \in \mathbb{R}$ such that $\frac{M_n - b_n}{a_n} \overset{d}{\Rightarrow} H$. Notation $F \in MDA(H)$.

Thm

The following conditions are necessary and sufficient for a distribution F to belong to the maximum domain of attraction of the three extremal type:

Fréchet: $F \in MDA(\Phi_\alpha)$ for some $\alpha > 0 \iff \bar{F}(x) = x^{-\alpha} L(x)$ for some L slowly varying at ∞. In this case $a_n = \bar{F}^{-1}\left(\frac{1}{n}\right)$ and $b_n = 0$.

Weibull: $F \in MDA(\Psi_\alpha)$ for some $\alpha > 0 \iff x_F < \infty$ and for all $t > 0$ $\bar{F}(x_F - t) = t^\alpha L(t)$ where L is slowly varying at 0. In this case $b_n = x_F$ and $a_n = x_F - \bar{F}^{-1}\left(\frac{1}{n}\right)$.

Gumbel: $F \in MDA(\Lambda) \iff \lim_{x \uparrow x_F} \frac{\bar{F}(x + t a(x))}{\bar{F}(x)} = e^{-t}$ for all $t \in \mathbb{R}$ for some function $a(x) > 0$.

In this case $b_n = \bar{F}^{-1}\left(\frac{1}{n}\right)$ and $a_n = a(b_n)$.

We only prove sufficiency above.

Proof

Fréchet

Let us first show that $n\overline{F}(a_n) \to 1$ as $n \to \infty$ where

$$a_n = \overline{F}^{-1}\left(\tfrac{1}{n}\right) = F^{-1}\left(1-\tfrac{1}{n}\right) = \inf\left\{t : 1-F(t) > 1-\tfrac{1}{n}\right\} = \inf\left\{t : \overline{F}(t) < \tfrac{1}{n}\right\}.$$

By definition, $\limsup_{n\to\infty} n\overline{F}(a_n) \leq 1$.

On the other hand, for any $t < a_n$, $\overline{F}(t) \geq \frac{1}{n}$ and so $n\overline{F}(t) \geq 1$.

In particular $\liminf_{n\to\infty} n\overline{F}(a_n -) \geq 1$.

By the assumption $\overline{F}(x) = x^{-\alpha}L(x)$, $\dfrac{\overline{F}(a_n x)}{\overline{F}(a_n)} \to x^{-\alpha}$. We take $x \nearrow 1$ to conclude by monotonicity that $\dfrac{\overline{F}(a_n -)}{\overline{F}(a_n)} \to 1$. Hence $n\overline{F}(a_n) \to 1$.

To prove convergence to Fréchet, let $x \leq 0$ first. Then

$F^n(a_n x) \leq F^n(0) \to 0$ as $n \to \infty$ since $F(0) < 1$, hence the limit of $\frac{M_n}{a_n}$ does not put any mass on $\mathbb{R}_-$.

Let $x > 0$. As we have just seen, $\overline{F}(a_n) \sim \frac{1}{n}$, hence $a_n \to \infty$ and

$$n\overline{F}(a_n x) \sim \frac{\overline{F}(a_n x)}{\overline{F}(a_n)} \to x^{-\alpha} \quad \text{as } n \to \infty \text{ by assumption.}$$

Then by Poisson approximation, $n\overline{F}(a_n x) \to x^{-\alpha}$ is equivalent to

$$\mathbb{P}\left(\frac{M_n}{a_n} < x\right) \to e^{-x^{-\alpha}}$$ which completes the proof.

Weibull

Similarly as before, one can show that $n\overline{F}(x_F - a_n) \to 1$ and $a_n \to 0$.

Then for $x < 0$, $n\overline{F}(a_n x + x_F) = \dfrac{\overline{F}(a_n x + x_F)}{\overline{F}(-a_n + x_F)} \to (-x)^{\alpha}$ as $n \to \infty$

By Poisson approximation, $\mathbb{P}\left(\dfrac{M_n - x_F}{a_n} < x\right) \to e^{-(-x)^{\alpha}}$ as $n \to \infty$.

By taking the $x \nearrow 0$ limit, we conclude the proof.

Gumbel

As before, $n\overline{F}(b_n) \to 1$ and $b_n \nearrow x_F$.

For any $x \in \mathbb{R}$ $\quad n\overline{F}(a(b_n)x + b_n) \sim \dfrac{\overline{F}(a(b_n)x + b_n)}{\overline{F}(b_n)} \to e^{-x}$ by assumption.

By Poisson approximation, $\mathbb{P}\left(\dfrac{M_n - b_n}{a(b_n)} < x\right) \to e^{-e^{-x}}$ as $n \to \infty$. $\square$

Examples

Fréchet $(a_n \asymp n^{1/\alpha})$

- Pareto distribution $\bar{F}(x) = \begin{cases} \left(\frac{x_m}{x}\right)^\alpha & \text{if } x \geq x_m \\ 1 & \text{if } x < x_m \end{cases}$ $x_m > 0$ scale, $\alpha > 0$ shape
- Cauchy
- stable laws with index $\alpha \in (0,2)$
- loggamma distribution density $f(x) = \frac{\lambda^\alpha (\ln x)^{\alpha-1} x^{-\lambda-1}}{\Gamma(\alpha)}$ $\alpha > 0$, $\lambda > 0$

Weibull $(a_n \asymp n^{-1/\alpha})$

- uniform
- power law behaviour at the right endpoint $\bar{F}(x) = K(x_F - x)^\alpha$ $\alpha > 0$

  $F \in MDA(\Psi_\alpha)$ $\bar{F}(x_F - a_n) = \frac{1}{n} \Rightarrow a_n = (nK)^{-1/\alpha}$
- beta distribution density $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ $x \in (0,1)$ $a, b > 0$

  $\bar{F}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b+1)}(1-x)^b$ as $x \nearrow 1 \Rightarrow F \in MDA(\Psi_b)$ by the prev. example

Gumbel

Remark

If the density of F exists, then one can choose $\frac{1}{a(x)} = \frac{f(x)}{\bar{F}(x)}$ to be the hazard rate. Reason: $\frac{1}{a(x)}$ is the conditional density of X_1 given that $\{X_1 > x\}$.

The condition for Gumbel: $\frac{\bar{F}(x + t a(x))}{\bar{F}(x)} = \mathbb{P}(X_1 - x > a(x)t \mid X_1 - x > 0)$

- exponential $\bar{F}(x) = e^{-\lambda x}$ $a(x) = \frac{1}{\lambda}$ $b_n = \frac{\log n}{\lambda}$ $a_n = \frac{1}{\lambda}$

  $\lambda M_n - \log n \xRightarrow{d} \Lambda$
- Weibull ((-1) times the usual one)

  $\bar{F}(x) = \exp(-x^\tau)$ $x \geq 0$ $\tau > 0$ $a(x) = \frac{x^{1-\tau}}{\tau}$

  $b_n = (\log n)^{1/\tau}$ $a_n = \frac{(\log n)^{1/\tau - 1}}{\tau}$ $\frac{\tau(M_n - (\log n)^{1/\tau})}{(\log n)^{1/\tau - 1}} \xRightarrow{d} \Lambda$ ($\tau = 1$ exponential)
- exponential behaviour at the finite right endpoint

  $\bar{F}(x) = K \exp\left(-\frac{\alpha}{x_F - x}\right)$ $x < x_F$ $\alpha, K > 0$ $a(x) = \frac{(x_F - x)^2}{\alpha}$ $x < x_F$

  $b_n = x_F - \frac{\alpha}{\log(Kn)}$ $a_n = \frac{\alpha}{(\log(Kn))^2}$

An example revisited

Let $X_1, X_2, \dots$ be an iid. sequence with distribution

$$\mathbb{P}(X_1 > x) = \mathbb{P}(X_1 < -x) = \frac{x^{-\alpha}}{2} \quad \text{for } x \geq 1 \quad \text{where } \alpha > 0.$$

Moments: $$\mathbb{E}(|X_1|^\beta) \begin{cases} < \infty & \text{if } 0 < \beta < \alpha \\ = \infty & \text{if } \beta \geq \alpha \end{cases}$$

$\boxed{\alpha > 2}$ $\mathbb{E}|X_1| < \infty$, $\mathbb{E}(|X_1|^2) < \infty$, LLN, CLT hold: $\frac{S_n}{n} \xrightarrow{\mathbb{P}} 0$, $\frac{S_n}{\sqrt{n}} \overset{d}{\Rightarrow} N(0,\sigma^2)$

$$\frac{M_n}{n^{1/\alpha}} \overset{d}{\Rightarrow} \Phi_\alpha \qquad 1/\alpha < 1/2$$

$\boxed{\alpha = 2}$ $\mathbb{E}|X_1| < \infty$, $\mathbb{E}(|X_1|^2) = \infty$, LLN $\frac{S_n}{n} \xrightarrow{\mathbb{P}} 0$, instead of CLT: $\frac{S_n}{\sqrt{n \log n}} \overset{d}{\Rightarrow} N(0,\sigma^2)$

$$\frac{M_n}{\sqrt{n}} \overset{d}{\Rightarrow} \Phi_2$$

$\boxed{1 < \alpha < 2}$ $\mathbb{E}|X_1| < \infty$, $\mathbb{E}(|X_1|^2) = \infty$, LLN $\frac{S_n}{n} \xrightarrow{\mathbb{P}} 0$, instead of CLT:

$$\frac{S_n}{n^{1/\alpha}} \overset{d}{\Rightarrow} \text{stable } \alpha. \quad 1/2 < 1/\alpha < 1, \quad \frac{M_n}{n^{1/\alpha}} \overset{d}{\Rightarrow} \Phi_\alpha$$

$\boxed{\alpha = 1}$ $\mathbb{E}|X_1| = \infty$, $\mathbb{E}(|X_1|^2) = \infty$, LLN fails, instead of CLT:

$$\frac{S_n}{n} \overset{d}{\Rightarrow} \text{Cauchy} \overset{d}{=} \text{stable } 1 \qquad \frac{M_n}{n} \overset{d}{\Rightarrow} \Phi_1$$

$\boxed{0 < \alpha < 1}$ $\mathbb{E}|X_1| = \infty$, $\mathbb{E}(|X_1|^2) = \infty$, LLN fails, instead of CLT:

$$\frac{S_n}{n^{1/\alpha}} \overset{d}{\Rightarrow} \text{stable } \alpha \quad 1/\alpha > 1 \qquad \frac{M_n}{n^{1/\alpha}} \overset{d}{\Rightarrow} \Phi_\alpha$$

The normal distribution

density: $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Lemma (tail asymptotics for the normal distr.)

For $\bar{F}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$ $\quad \bar{F}(x) = \frac{1}{x}\varphi(x)\big(1+o(1)\big)$ as $x\to\infty$.

Proof: by L'Hospital's rule $\lim\limits_{x\to\infty} \frac{\bar{F}(x)}{\frac{1}{x}\varphi(x)} = \lim\limits_{x\to\infty} \frac{-\varphi(x)}{-\frac{1}{x^2}\varphi(x)-\varphi(x)} = \lim\limits_{x\to\infty}\frac{1}{\frac{1}{x^2}+1} = 1$ □

Hazard rate: $\frac{1}{a(x)} = \frac{\varphi(x)}{\bar{F}(x)} = x\big(1+o(1)\big)$ as $x\to\infty$

Hence $a(x) = \frac{1}{x}\big(1+o(1)\big)$ as $x\to\infty$.

The condition for Gumbel:

$$\frac{\bar{F}\big(x+a(x)t\big)}{\bar{F}(x)} = \frac{x}{x+a(x)t}\,\frac{e^{-\frac{(x+a(x)t)^2}{2}}}{e^{-\frac{x^2}{2}}}\big(1+o(1)\big) = \frac{x}{x+\frac{t}{x}}\,\frac{e^{-\frac{(x+\frac{t}{x})^2}{2}}}{e^{-\frac{x^2}{2}}}\big(1+o(1)\big)$$

$$= \frac{x}{x+\frac{t}{x}}\, e^{-t-\frac{t^2}{2x^2}}\big(1+o(1)\big) \to e^{-t} \quad \text{as } x\to\infty$$

To find the normalization, one solves $\bar{F}(b_n) = \frac{1}{n}$, i.e.

$\frac{1}{\sqrt{2\pi}\, b_n} e^{-\frac{b_n^2}{2}}\big(1+o(1)\big) = \frac{1}{n}$ since $b_n\to\infty$.

$$\frac{1}{2}\log(2\pi) + \log b_n + \frac{1}{2}b_n^2 = \log n + o(1)$$

The largest term on the LHS is $\frac{1}{2}b_n^2$, hence $b_n = \sqrt{2\log n}\big(1+o(1)\big)$

We already see at this point that $a_n = a(b_n) = \frac{1}{\sqrt{2\log n}}\big(1+o(1)\big)$

One can write $b_n = \sqrt{2\log n} + c_n$, for which $c_n = o(\sqrt{\log n})$ and

$$\frac{1}{2}\log(2\pi) + \underbrace{\log\left(\sqrt{2\log n}+c_n\right)}_{\frac{1}{2}\lg 2+\frac{1}{2}(\log\log n+o(\log\log n))} + \underbrace{\frac{1}{2}\left(\sqrt{2\log n}+c_n\right)^2}_{\log n+\sqrt{2\log n}\,c_n+\frac{c_n^2}{2}} = \log n$$

Since $c_n = o(\sqrt{\log n})$, we solve

$$\frac{1}{2}\log(4\pi) + \frac{1}{2}\log\log n + \sqrt{2\log n}\, c_n = 0 \quad\Rightarrow\quad c_n = -\frac{\log\log n+\log(4\pi)}{2\sqrt{2\log n}}$$

As a conclusion, $\sqrt{2\log n}\left(M_n - \sqrt{2\log n} + \frac{\log\log n+\log(4\pi)}{2\sqrt{2\log n}}\right) \xrightarrow{d} \Lambda$.