# Maximum domains of attraction

## Examples

① Let $X_1, X_2, \ldots$ iid. Cauchy with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$

Then $\bar{F}(x) \sim \frac{1}{\pi x}$ as $x \to \infty$.

$$\mathbb{P}\left(M_n < \frac{nx}{\pi}\right) = \left(1 - \bar{F}\left(\frac{nx}{\pi}\right)\right)^n = \left(1 - \frac{1}{nx} + o\left(\frac{1}{n}\right)\right)^n \to e^{-\frac{1}{x}} = \Phi_1(x)$$

$$\frac{\pi}{n} M_n \overset{d}{\Rightarrow} \Phi_1$$

② Let $X_1, X_2, \ldots$ iid. $EXP(1)$. Then $\bar{F}(x) = e^{-x}$.

$$\mathbb{P}(M_n - \ln n < x) = \left(1 - \bar{F}(x + \ln n)\right)^n = \left(1 - \frac{e^{-x}}{n}\right)^n \to e^{-e^{-x}} = \Lambda(x)$$

$$M_n - \ln n \overset{d}{\Rightarrow} \Lambda$$

## Def

The distribution function $F$ belongs to the <u>maximum domain of attraction</u> of the extreme value distribution $H$, if there are $a_n > 0$ and $b_n \in \mathbb{R}$ such that $\frac{M_n - b_n}{a_n} \overset{d}{\Rightarrow} H$. Notation $F \in MDA(H)$.

## Thm

The following conditions are necessary and sufficient for a distribution $F$ to belong to the maximum domain of attraction of the three extremal type:

<u>Fréchet</u>: $F \in MDA(\Phi_\alpha)$ for some $\alpha > 0 \iff \bar{F}(x) = x^{-\alpha} L(x)$ for some $L$ slowly varying at $\infty$. In this case $a_n = \bar{F}^{-1}\left(\frac{1}{n}\right)$ and $b_n = 0$.

<u>Weibull</u>: $F \in MDA(\Psi_\alpha)$ for some $\alpha > 0 \iff x_F < \infty$ and for all $t > 0$ $\bar{F}(x_F - t) = t^\alpha L(t)$ where $L$ is slowly varying at $0$. In this case $b_n = x_F$ and $a_n = x_F - \bar{F}^{-1}\left(\frac{1}{n}\right)$.

<u>Gumbel</u>: $F \in MDA(\Lambda) \iff \lim\limits_{x \uparrow x_F} \frac{\bar{F}(x + t a(x))}{\bar{F}(x)} = e^{-t}$ for all $t \in \mathbb{R}$

for some function $a(x) > 0$.
In this case $b_n = \bar{F}^{-1}\left(\frac{1}{n}\right)$ and $a_n = a(b_n)$.

We only prove sufficiency above.

## Proof

### Fréchet

We assume $\bar{F}(x) = x^{-\alpha} L(x)$ as $x \to \infty$ and we set $a_n = \bar{F}^{-1}(\frac{1}{n})$.

Then $a_n \to \infty$. We will show first that $n\bar{F}(a_n) \to 1$ as $n \to \infty$.

By definition, $a_n = \bar{F}^{-1}(\frac{1}{n}) = \inf\{t : \bar{F}(t) < \frac{1}{n}\} = \sup\{t : \bar{F}(t) \geq \frac{1}{n}\} = \sup\{t : n\bar{F}(t) \geq 1\}$

By the left-continuity of $\bar{F}$, it yields that $\liminf_{n\to\infty} n\bar{F}(a_n) \geq 1$.

On the other hand, $\limsup_{n\to\infty} n\bar{F}(a_n+) \leq 1$.

By the assumption $\bar{F}(x) = x^{-\alpha} L(x)$, we have for any $x > 0$ that

$$\frac{\bar{F}(xa_n)}{\bar{F}(a_n)} = \frac{x^{-\alpha} a_n^{-\alpha} L(xa_n)}{a_n^{-\alpha} \quad L(a_n)} \longrightarrow x^{-\alpha} \quad \text{as } n \to \infty \quad \text{since } a_n \to \infty.$$

By taking $x \searrow 1$, we conclude by monotonicity that $\frac{\bar{F}(a_n+)}{\bar{F}(a_n)} \to 1$,

hence $n\bar{F}(a_n) \to 1$.

To prove convergence to Fréchet, let $x \leq 0$ first. Then

$F^n(a_n x) \leq F^n(0) \to 0$ as $n \to \infty$ since $F(0) < 1$, hence

the limit of $M_n/a_n$ does not put any mass on $\mathbb{R}_-$.

Let $x > 0$. As we have just seen, $\bar{F}(a_n) \sim \frac{1}{n}$ and $a_n \to \infty$,

$$n\bar{F}(a_n x) \sim \frac{\bar{F}(a_n x)}{\bar{F}(a_n)} \longrightarrow x^{-\alpha} \quad \text{as } n \to \infty$$

Then by Poisson approximation, $n\bar{F}(a_n x) \to x^{-\alpha}$ is equivalent to

$\mathbb{P}\left(\frac{M_n}{a_n} < x\right) \longrightarrow e^{-x^{-\alpha}}$ which completes the proof.

### Weibull

Similarly as before, with $\bar{F}(x_F - t) = t^{\alpha} L(t)$ and $a_n = x_F - \bar{F}^{-1}(\frac{1}{n})$

we have $a_n \to 0$ and $n\bar{F}(x_F - a_n) \to 1$ as $n \to \infty$.

For $x < 0$, $n\bar{F}(a_n x + x_F) \sim \frac{\bar{F}(x_F - a_n(-x))}{\bar{F}(x_F - a_n)} = \frac{a_n^{\alpha}(-x)^{\alpha} L(a_n(-x))}{a_n^{\alpha} \; L(a_n)} \to (-x)^{\alpha}$ as $n \to \infty$

By Poisson approximation, $\mathbb{P}\left(\frac{M_n - x_F}{a_n} < x\right) \longrightarrow e^{-(-x)^{\alpha}}$ as $n \to \infty$.

### Gumbel

As before with $b_n = \bar{F}^{-1}(\frac{1}{n})$ and assuming $\lim_{x \nearrow x_F} \frac{\bar{F}(x + t a(x))}{\bar{F}(x)} = e^{-t}$,

we have $b_n \nearrow x_F$ and $n\bar{F}(b_n) \to 1$ as $n \to \infty$.

For any $x \in \mathbb{R}$ $n\bar{F}(a(b_n)x + b_n) \sim \frac{\bar{F}(b_n + x a(b_n))}{\bar{F}(b_n)} \to e^{-x}$ as $n \to \infty$ by assumption.

By Poisson approximation, $\mathbb{P}\left(\frac{M_n - b_n}{a(b_n)} < x\right) \longrightarrow e^{-e^{-x}}$ as $n \to \infty$.

## Examples

### Fréchet ($a_n \asymp n^{1/\alpha}$)

- Pareto distribution $\bar{F}(x) = \begin{cases} \left(\frac{x_m}{x}\right)^\alpha & \text{if } x \geq x_m \\ 1 & \text{if } x < x_m \end{cases}$ $x_m > 0$ scale, $\alpha > 0$ shape
- Cauchy
- stable laws with index $\alpha \in (0,2)$
- loggamma distribution density $f(x) = \dfrac{\lambda^\alpha (\ln x)^{\alpha-1} x^{-\lambda-1}}{\Gamma(\alpha)}$ $\alpha > 0$, $\lambda > 0$

### Weibull ($a_n \asymp n^{-1/\alpha}$)

- uniform
- power law behaviour at the right endpoint $\bar{F}(x) = K(x_F - x)^\alpha$ $\alpha > 0$

  $F \in MDA(\Psi_\alpha)$ $\bar{F}(x_F - a_n) = \frac{1}{n} \Rightarrow a_n = (nK)^{-1/\alpha}$
- beta distribution density $f(x) = \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ $x \in (0,1)$ $a,b > 0$

  $\bar{F}(x) = \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b+1)}(1-x)^b$ as $x \nearrow 1$ $\Rightarrow F \in MDA(\Psi_b)$ by the prev. example

### Gumbel

**Remark**

If the density of $F$ exists, then one can choose $\frac{1}{a(x)} = \frac{f(x)}{\bar{F}(x)}$ to be the hazard rate. Reason: $\frac{1}{a(x)}$ is the conditional density of $X_1$ given that $\{X_1 > x\}$. The condition for Gumbel: $\dfrac{\bar{F}(x + t\,a(x))}{\bar{F}(x)} = \mathbb{P}(X_1 - x > a(x)t \mid X_1 - x > 0)$

- exponential $\bar{F}(x) = e^{-\lambda x}$ $a(x) = \frac{1}{\lambda}$ $b_n = \frac{\log n}{\lambda}$ $a_n = \frac{1}{\lambda}$

  $\lambda M_n - \log n \overset{d}{\Rightarrow} \Lambda$
- Weibull ((-1) times the usual one)

  $\bar{F}(x) = \exp(-x^\tau)$ $x \geq 0$ $\tau > 0$ $a(x) = \dfrac{x^{1-\tau}}{\tau}$

  $b_n = (\log n)^{1/\tau}$ $a_n = \dfrac{(\log n)^{1/\tau - 1}}{\tau}$ $\dfrac{\tau\left(M_n - (\log n)^{1/\tau}\right)}{(\log n)^{1/\tau - 1}} \overset{d}{\Rightarrow} \Lambda$ ($\tau = 1$ exponential)
- exponential behaviour at the finite right endpoint

  $\bar{F}(x) = K \exp\left(-\dfrac{\alpha}{x_F - x}\right)$ $x < x_F$ $\alpha, K > 0$ $a(x) = \dfrac{(x_F - x)^2}{\alpha}$ $x < x_F$

  $b_n = x_F - \dfrac{\alpha}{\log(Kn)}$ $a_n = \dfrac{\alpha}{(\log(Kn))^2}$

## An example revisited

Let $X_1, X_2, \dots$ be an iid. sequence with distribution

$$\mathbb{P}(X_1 > x) = \mathbb{P}(X_1 < -x) = \frac{x^{-\alpha}}{2} \quad \text{for } x \geq 1 \quad \text{where } \alpha > 0.$$

Moments: $\mathbb{E}(|X_1|^\beta) \begin{cases} < \infty & \text{if } 0<\beta<\alpha \\ = \infty & \text{if } \beta \geq \alpha \end{cases}$

$\boxed{\alpha > 2}$ $\mathbb{E}|X_1| < \infty$, $\mathbb{E}|X_1|^2 < \infty$, LLN, CLT hold: $\frac{S_n}{n} \xrightarrow{\mathbb{P}} 0$, $\frac{S_n}{\sqrt{n}} \xRightarrow{d} N(0,\sigma^2)$

$\frac{M_n}{n^{1/\alpha}} \xRightarrow{d} \Phi_\alpha$ $\quad 1/\alpha < 1/2$

$\boxed{\alpha = 2}$ $\mathbb{E}|X_1| < \infty$, $\mathbb{E}(|X_1|^2) = \infty$, LLN $\frac{S_n}{n} \xrightarrow{\mathbb{P}} 0$; instead of CLT: $\frac{S_n}{\sqrt{n \log n}} \xRightarrow{d} N(0,\sigma^2)$

$\frac{M_n}{\sqrt{n}} \xRightarrow{d} \Phi_2$

$\boxed{1 < \alpha < 2}$ $\mathbb{E}|X_1| < \infty$, $\mathbb{E}(|X_1|^2) = \infty$, LLN $\frac{S_n}{n} \xrightarrow{\mathbb{P}} 0$, instead of CLT:

$\frac{S_n}{n^{1/\alpha}} \xRightarrow{d}$ stable $\alpha$. $\quad 1/2 < 1/\alpha < 1$, $\quad \frac{M_n}{n^{1/\alpha}} \xRightarrow{d} \Phi_\alpha$

$\boxed{\alpha = 1}$ $\mathbb{E}|X_1| = \infty$, $\mathbb{E}(|X_1|^2) = \infty$, LLN fails, instead of CLT:

$\frac{S_n}{n} \xRightarrow{d}$ Cauchy $\stackrel{d}{=}$ stable 1 $\qquad \frac{M_n}{n} \xRightarrow{d} \Phi_1$

$\boxed{0 < \alpha < 1}$ $\mathbb{E}|X_1| = \infty$, $\mathbb{E}(|X_1|^2) = \infty$, LLN fails, instead of CLT:

$\frac{S_n}{n^{1/\alpha}} \xRightarrow{d}$ stable $\alpha$ $\quad 1/\alpha > 1$ $\qquad \frac{M_n}{n^{1/\alpha}} \xRightarrow{d} \Phi_\alpha$

## The normal distribution

density: $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

**Lemma** (tail asymptotics for the normal distr.)

For $\bar{F}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy$ $\quad \bar{F}(x) = \frac{1}{x}\varphi(x)(1+o(1))$ as $x\to\infty$.

**Proof**: by L'Hospital's rule $\lim\limits_{x\to\infty} \frac{\bar{F}(x)}{\frac{1}{x}\varphi(x)} = \lim\limits_{x\to\infty} \frac{-\varphi(x)}{-\frac{1}{x^2}\varphi(x) - \varphi(x)} = \lim\limits_{x\to\infty} \frac{1}{\frac{1}{x^2}+1} = 1$ $\square$

Hazard rate: $\frac{1}{a(x)} = \frac{\varphi(x)}{\bar{F}(x)} = x(1+o(1))$ as $x\to\infty$

Hence $a(x) = \frac{1}{x}(1+o(1))$ as $x\to\infty$.

The condition for Gumbel:

$$\frac{\bar{F}(x+a(x)t)}{\bar{F}(x)} = \frac{x}{x+a(x)t}\,\frac{e^{-\frac{(x+a(x)t)^2}{2}}}{e^{-x^2/2}}(1+o(1)) = \frac{x}{x+\frac{t}{x}}\,\frac{e^{-\frac{(x+\frac{t}{x})^2}{2}}}{e^{-\frac{x^2}{2}}}(1+o(1))$$

$$= \frac{x}{x+\frac{t}{x}} e^{-t-\frac{t^2}{2x^2}}(1+o(1)) \to e^{-t} \quad \text{as } x\to\infty$$

To find the normalization, one solves $\bar{F}(b_n) = \frac{1}{n}$, i.e.

$\frac{1}{\sqrt{2\pi}\,b_n} e^{-\frac{b_n^2}{2}}(1+o(1)) = \frac{1}{n}$ since $b_n\to\infty$.

$$\frac{1}{2}\log(2\pi) + \log b_n + \frac{1}{2}b_n^2 = \log n + o(1)$$

The largest term on the LHS is $\frac{1}{2}b_n^2$, hence $b_n = \sqrt{2\log n}\,(1+o(1))$

We already see at this point that $a_n = a(b_n) = \frac{1}{\sqrt{2\log n}}(1+o(1))$

One can write $b_n = \sqrt{2\log n} + c_n$, for which $c_n = o(\sqrt{\log n})$ and

$$\frac{1}{2}\log(2\pi) + \underbrace{\log\left(\sqrt{2\log n}+c_n\right)}_{\frac{1}{2}\lg 2 + \frac{1}{2}(\log\log n + o(\log\log n))} + \underbrace{\frac{1}{2}\left(\sqrt{2\log n}+c_n\right)^2}_{\log n + \sqrt{2\log n}\,c_n + \frac{c_n^2}{2}} = \log n$$

Since $c_n = o(\sqrt{\log n})$, we solve

$$\frac{1}{2}\log(4\pi) + \frac{1}{2}\log\log n + \sqrt{2\log n}\,c_n = 0 \implies c_n = -\frac{\log\log n + \log(4\pi)}{2\sqrt{2\log n}}$$

As a conclusion, $\sqrt{2\log n}\left(M_n - \sqrt{2\log n} + \frac{\log\log n + \log(4\pi)}{2\sqrt{2\log n}}\right) \xrightarrow{d} \Lambda$.

# Hazard rate

Also called as: hazard function, failure rate, force of mortality

Assume that $X$ is a (non-negative) random variable with distribution function $F(x) = \mathbb{P}(X < x)$, $\bar{F}(x) = 1 - F(x)$ and density $f(x) = F'(x)$.

Let
$$h(x) = \frac{f(x)}{\bar{F}(x)} = -\frac{d}{dx}\log(\bar{F}(x)) = \lim_{\varepsilon \to 0} \frac{\mathbb{P}(X \in (x, x+\varepsilon) \mid X \geq x)}{\varepsilon}$$

be the hazard rate.

## Examples

- $EXP(\lambda)$ $\quad h(x) = \lambda \quad x \geq 0$
- $U[0,1]$ $\quad h(x) = \frac{1}{1-x} \quad x \in [0,1]$
- $N(0,1)$ $\quad h(x) \sim x$ as $x \to \infty$
- Pareto : $\bar{F}(x) = x^{-\alpha}$, $f(x) = \alpha x^{-\alpha-1}$ $\quad h(x) = \frac{\alpha}{x}$

## Proposition

① For a distribution with tail probability function $\bar{F}(x)$ and hazard rate $h(x)$, we have
$$\bar{F}(x) = e^{-\int_0^x h(y)\,dy}$$

② A measurable function $h$ is the hazard rate of some distribution if and only if $h(x) \geq 0$ for all $x \geq 0$ and $\int_0^\infty h(x)\,dx = \infty$.

## Proof:

①
$$\int_0^x h(y)\,dy = \int_0^x \frac{f(y)}{\bar{F}(y)}\,dy = \left[-\log(\bar{F}(y))\right]_0^x = -\log \bar{F}(x)$$

② The conditions on $h$ guarantee that the distribution given by the formula in ① is well-defined.

Cumulative hazard function : $H(x) = \int_0^x h(y)\,dy = -\log \bar{F}(x)$

## Condition for Gumbel

$$\mathbb{P}(X > x + ta(x) \mid X > x) = \frac{\bar{F}(x + ta(x))}{\bar{F}(x)} = e^{-\int_x^{x+ta(x)} h(y)\,dy} \to e^{-t}$$

as $x \to x_F$, that is $\lim_{x \to x_F} \int_x^{x+ta(x)} h(y)\,dy = t$

Possible choices of $a(x)$:

$$a(x) = \frac{1}{h(x)} \quad \text{or} \quad a(x) = \frac{1}{\bar{F}(x)} \int_x^{x_F} \bar{F}(y)\,dy$$