LECTURE 1

In this lecture we introduce the basic concepts used throughout the semester.

We deal with only finite dimensional Euclidean spaces. We regard an n-dimensional Euclidean spaces as an affine space whose vectors are the elements of the n-dimensional vector space $\mathbb{R}^n$ over the set of real numbers. Fixing an arbitrary point of an affine space, the elements of the corresponding vector space and the points of the space can be identified in a natural way, in which a point is associated to the vector that moves the fixed point to this one. In this case the fixed point is usually called *origin*. As it often appears in the literature, during the term we identify the Euclidean space with the vector space $\mathbb{R}^n$ (in high school language: we identify points and their position vectors). We will usually denote the points/vectors of the space $\mathbb{R}^n$ by small Latin letters, while its subsets by capital Latin letters.

We denote the usual inner (scalar) product of $\mathbb{R}^n$ by $\langle .,. \rangle$. The length $||v||$ of a vector $v \in \mathbb{R}^n$ is the quantity $\sqrt{\langle v, v \rangle}$. For the coordinates of the vector/point v in the standard orthonormal basis of $\mathbb{R}^n$ we use the notation $v = (v_1, v_2, \ldots, v_n)$. We denote the origin by o. The *distance* of the points $p = (x_1, x_2, \ldots, x_n)$ and $q = (x'_1, x'_2, \ldots, x'_n)$, denoted by $\operatorname{dist}(p, q)$, is the quantity $\sqrt{\sum_{i=1}^{n} (x'_i - x_i)^2}$, which coincides with the value of $||q - p||$. The *interior*, *boundary*, *closure* and *cardinality* of a set $X \subseteq \mathbb{R}^n$ will be denoted by $\operatorname{int}(X), \operatorname{bd}(X), \operatorname{cl}(X), |X|$, respectively.

Definition 1. *Let V_1 and V_2 be two point sets, and $\lambda \in \mathbb{R}$. Then*

$$V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$$

is called the Minkowski sum of the two sets*, and*

$$\lambda V_1 = \{\lambda v_1 : v_1 \in V_1\}$$

the multiple *of V_1 by λ.*

Definition 2. *Let $p \in \mathbb{R}^n$ be an arbitrary point, and L and arbitrary (linear) subspace in the vector space $\mathbb{R}^n$. Then the set $p + L \subseteq \mathbb{R}^n$ is called an* affine subspace *of the space $\mathbb{R}^n$.*

The next remark is a straightforward consequence of the properties of linear subspaces.

Remark 1. *Let $p, q \in \mathbb{R}^n$ and let L, L' be linear subspaces in $\mathbb{R}^n$. Then $p + L = q + L'$ is satisfied if and only if $L = L'$ and $q \in p + L$.*

Proof. Assume that $p + L = q + L'$. Then $L = (q - p) + L'$ by the definition of Minkowski sum, which yields, in particular, that $q - p \in L$, from which we have $q \in p + L$. But as linear subspaces are closed with respect to addition, $q - p \in L$ implies $(q - p) + L = L$, from which $(q - p) + L = (q - p) + L'$, yielding $L = L'$. On the other hand, if $q \in p + L$, then $(q - p) \in L \Longrightarrow (q - p) + L = L \Longrightarrow q + L = p + L$. □

Theorem 1. *A nonempty intersection of affine subspaces is an affine subspace.*

Proof. Consider the affine subspaces A_i $(i \in I)$, where I is an arbitrary index set. Let $A = \bigcap_{i \in I} A_i$. Consider a point $p \in A$. Then, due to the previous remark, for any $i \in I$ we have $A_i = p + L_i$ for some suitable linear subspace L_i of $\mathbb{R}^n$. The intersection of linear subspaces is a linear subspace, and thus, $L = \bigcap_{i \in I} L_i$ is a linear subspace. On the other hand, we clearly have $A = p + L$, from which the assertion follows. □

By the *dimension* of an affine subspace we mean the dimension of the corresponding linear subspace. We call the 0-, 1-, 2-, $(n-1)$-dimensional subspaces *points, lines, planes* and *hyperplanes*. A k-dimensional affine subspace may also be called a k-flat.

The next property readily follows from the definition of affine subspaces and the properties of inner product.

Remark 2. *If $u \in \mathbb{R}^n$ and $t \in \mathbb{R}$ arbitrary, then the set $\{v \in \mathbb{R}^n : \langle v, u \rangle = t\}$ is a hyperplane. Furthermore, for any hyperplane H there is some vector $u \in \mathbb{R}^n$ and scalar $t \in \mathbb{R}$ for which $H = \{v \in \mathbb{R}^n : \langle v, u \rangle = t\}$.*

Since inner product is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}$, the previous remark implies that for any hyperplane H decomposes the space into two connected, open components, which we call *open half spaces*. The unions of open half spaces with the bounding hyperplane we call *closed half spaces*.

Definition 3. *Let $G_1 = p_1 + L_1$ and $G_2 = p_2 + L_2$ be affine subspaces. If for any vectors $v_1 \in L_1$, $v_2 \in L_2$ we have $\langle v_1, v_2 \rangle = 0$, then we say that G_1 and G_2 are* perpendicular *or* orthogonal. *Two affine subspaces are* parallel, *if they can be written in the form $p_1 + L$ and $p_2 + L$, where L is a linear subspace.*

Definition 4. *Let* $X \subset \mathbb{R}^n$ *be a nonempty set. Then the* affine hull *of* X*, denoted by* $\operatorname{aff}(X)$*, is defined as the intersection of all affine subspaces containing* X*. The* linear hull *of* X *is defined as the affine hull* $\operatorname{aff}(X \cup \{o\})$*. We denote the linear hull of* X *by* $\operatorname{lin}(X)$*. The* relative interior *and* relative boundary *of* X *is defined as the interior and boundary of* X*, respectively, with respect to the induced topology in* $\operatorname{aff}(X)$*. We denote them by* $\operatorname{relint}(X)$ *and* $\operatorname{relbd}(X)$*, respectively.*

We remark that by Theorem 1, the affine hull of a set is an affine subspace.

Definition 5. *A point set* X *is called* affinely independent *if for any* $x \in X$ *we have* $\operatorname{aff}(X \setminus \{x\}) \neq \operatorname{aff} X$*. The points sets that are not affinely independent are called* affinely dependent*.*

Definition 6. *Let* $p_1, p_2, \ldots, p_k \in \mathbb{R}^n$ *finitely many points, and let* $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ *be real numbers satisfying* $\sum_{i=1}^{k} \lambda_i = 1$*. Then the point* $\sum_{i=1}^{k} \lambda_i p_i$ *is called an* affine combination *of the points* $p_1, p_2, \ldots, p_k$*.*

Proposition 1. *The affine hull of a set* X *is the set of the affine combinations of all finite point sets from* X*.*

Proof. Let Y denote the set of all affine combinations of finitely many points in X, and let $p \in X$ be an arbitrary point. Consider the points $p_1 = p, p_2, \ldots, p_k \in X$ and numbers $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ for which $\sum_{i=1}^{k} \lambda_i = 1$ is satisfied. According to our conditions:

$$\sum_{i=1}^{k} \lambda_i p_i = p_1 + \sum_{i=1}^{k} \lambda_i (p_i - p_1).$$

Thus the affine combination can be written as a translate of the point p with a linear combination of the vectors $p_i - p$. Hence, if L denotes the linear subspace formed by the linear combinations of the vectors $q - p$, $q \in X$, then $Y = p + L$. As it is clearly an affine subspace, we have $\operatorname{aff}(X) \subseteq Y$.

On the other hand, if an affine subspace contains X, then it can be written in the form $p + L$ for some linear subspace L. The subspace L contains all vectors of the form $q - p$, $q \in X$, and thus it contains their linear combinations as well. Hence, $p + L$ contains all affine combinations of points of X in the case that p is one of the points. Since any k-point affine combination is also a $(k+1)$-point affine combination in which one of the points is p, we have that $p + L$ contains all affine combinations of the points of X. Thus, $Y \subseteq p + L$, implying $Y \subseteq \text{aff}(X)$. $\square$

Corollary 1. *A point set X is affinely independent if and only if there is no point of X that can be written as an affine combination of some other points from X.*

Theorem 2. *Let $X = \{p_1, p_2, \ldots, p_k\} \subset \mathbb{R}^n$. Then X is affinely independent if and only if $\sum_{i=1}^{k} \lambda_i p_i = 0$ and $\sum_{i=1}^{k} \lambda_i = 0$ implies $\lambda_i = 0$ for all values of i.*

Proof. Assume that a pont, say p_k, can be written as an affine combination of the other points; that is, $p_k = \sum_{i=1}^{k-1} \lambda_i p_i$, where $\sum_{i=1}^{k-1} \lambda_i = 1$. Then, setting $\lambda_k = -1$, we have $0 = \sum_{i=1}^{k} \lambda_i p_i$ and $\sum_{i=1}^{k} \lambda_i = 0$.

On the other hand, assume that for some values of the coefficients λ_i, not all of them zero, we have $0 = \sum_{i=1}^{k} \lambda_i p_i$ and $\sum_{i=1}^{k-1} \lambda_i = 0$. Without loss of generality, we may assume that $\lambda_k \neq 0$. For any $1 \leq i \leq k-1$, let $\lambda'_i = -\frac{\lambda_i}{\lambda_k}$. Then

$$\sum_{i=1}^{k-1} \lambda'_i = -\frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_k} = -\frac{-\lambda_k}{\lambda_k} = 1,$$

and

$$\sum_{i=1}^{k-1} \lambda'_i p_i = -\frac{1}{\lambda_k} \sum_{i=1}^{k-1} \lambda_i p_i = -\frac{1}{\lambda_k} (-\lambda_k p_k) = p_k,$$

and the point set is affinely dependent. $\square$ $\square$

Corollary 2. *If $X \subset \mathbb{R}^n$ is affinely independent, then every point of $\text{aff}(X)$ can be uniquely written as an affine combination of some points in X.*

Theorem 3. *If $|X| \geq n + 2$, then X is affinely dependent.*

Proof. Assume that $p_1, p_2, \ldots, p_{n+2} \in X$. Consider the vectors $p_2 - p_1, \ldots, p_{n+2} - p_1$. Since the n-dimensional Euclidean space is an n-dimensional vector space, the above vectors are linearly dependent, that is one of them, say $p_{n+2} - p_1$, can be written as a linear combination of the other vectors: $p_{n+2} - p_1 = \sum_{i=2}^{n+1} \lambda_i (p_i - p_1)$. Let $\lambda_1 = 1 - \sum_{i=2}^{n+1} \lambda_i$.

Then clearly $\sum\limits_{i=1}^{n+1} \lambda_i = 1$. On the other hand,

$$p_{n+2} = p_1 + \sum_{i=2}^{n+1} \lambda_i (p_i - p_1) = \left(1 - \sum_{i=2}^{n+1} \lambda_i\right) p_1 + \sum_{i=2}^{n+1} \lambda_i p_i = \sum_{i=1}^{n+1} \lambda_i p_i,$$

that is, X is affinely dependent. □

Corollary 3. *Every affine subspace of the space $\mathbb{R}^n$ is the affine hull of a most $n+1$ points.*

We continue with a new topic.

Definition 7. *Let $p_1, p_2, \dots, p_k \in \mathbb{R}^n$. If a point p can be written in the form $\sum\limits_{i=1}^{k} \lambda_i p_i$, $\sum_{i=1}^{k} \lambda_i = 1$, where $\lambda_i \geq 0$ for all is, then we say that p is a* convex combination *of the points $p_1, p_2, \dots, p_k$.*

Definition 8. *The set of the convex combinations of the points $p, q \in \mathbb{R}^n$ is called the* closed segment *with endpoints p and q.If $p \neq q$, then the set $[p,q] \setminus \{p,q\}$ is called the* open segment *with endpoints p and q, and it is denoted by (p,q).*

Definition 9. *Let $K \subseteq \mathbb{R}^n$. The set K is called* convex, *if for arbitrary $p, q \in K$ we have $[p,q] \subseteq K$.*