LECTURE 10: POLYTOPES, POLYHEDRAL SETS, THEIR FACE STRUCTURES

We continue our investigation of convex polytopes.

Theorem 1. *Every bounded polyhedral set is a convex polytope.*

Proof. Every bounded polyhedral set $P \subset \mathbb{R}^n$ is a compact, convex set. Thus, by the Krein-Milman Theorem, it is sufficient to show that P has finitely many extremal points. We prove this by induction on the dimension n. If $n = 1$, then every compact, convex set (in particular, P) is a closed segment with two extremal points, the endpoints of the segment. Thus, for $n = 1$ the statement holds. Now, let P be an n-dimensional polyhedral set, and let $H_1, \ldots, H_k$ be the hyperplanes bounding the closed half spaces defining P.

Let $x \in \operatorname{ext}(P)$. If $x \in P$ and $x \notin H_i$ for any i, then, by the continuity of linear functionals, $x \in \operatorname{int}(P)$, implying $x \notin \operatorname{ext}(P)$. Thus, we can assume that $x \in H_i$ for some value of i. By Theorem 1 of the fifth lecture, for any closed, convex set K and any supporting hyperplane H of K, we have $\operatorname{ext}(K) \cap H = \operatorname{ext}(K \cap H)$. This yields that $\operatorname{ext}(H_i \cap P) = \operatorname{ext}(P) \cap H_i$. But, by the induction hypothesis, $|\operatorname{ext}(H_i \cap P)| < \infty$, implying $|\operatorname{ext}(P)| \leq \sum_{i=1}^{m} |\operatorname{ext}(H_i \cap P)| < \infty$. □

Let us recall the definition of algebraic lattice.

Definition 1. *Let $(A, \leq)$ be a partially ordered set. If, for any $a_1, a_2, \ldots, a_k \in A$ there is a $c \in A$ such that $c \leq a_i$ for every value of i, and if $d \in A$, $d \leq a_i$ for every i implies that $d \leq c$, then we say that c is the* infimum *of $a_1, \ldots, a_k$. One can define the* supremum *of $a_1, \ldots, a_k$ similarly. If for any $a, b \in A$, a and b has an infimum and a supremum, we say that $(A, \leq)$ is an (algebraic)* lattice.

Definition 2. *Assume that $(A \leq)$ is a lattice with a minimal element, denoted by 0, that is, assume that there is an element $0 \in A$ such that $0 \leq a$ for all $a \in A$. We say that $a \in A$, $a \neq 0$ is an* atom, *if $b \in A$, and $b \leq a$ implies $b = a$ or $b = 0$. We say that $(A, \leq)$ is* atomic, *if for every $b \in A$, $b \neq 0$ there is an atom $a \in A$ satisfying $a \leq b$. We say that $(A, \leq)$ is* atomistic, *if every element of A is the supremum of some atoms in A.*

Theorem 2. *Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope, and let $\mathcal{F}$ the family consisting of the faces of P (including the empty set),* *and also P. Then $\mathcal{F}$ is a lattice with respect to the partial order defined by the containment relation. This lattice is atomic and atomistic, and its atoms are the vertices of P.*

Proof. Let $F \in \mathcal{F}$. Then the infimum and the supremum of $\emptyset$ and F is $\emptyset$ and F, respectively, and the infimum and the supremum of P and F are F and P, respectively. Now, let F_1 and F_2 be proper faces of P. We have seen that $F = F_1 \cap F_2$ is a face of P. Clearly, for any $F' \in \mathcal{F}$ with $F' \subseteq F_1$ and $F' \subseteq F_2$, we have $F' \subseteq F$, and thus, F is the infimum of F_1 and F_2.

We show that F_1 and F_2 has a supremum. Indeed, if there is no proper face of P that contains both F_1 and F_2, then, clearly, P is the supremum of F_1 and F_2. If there is a proper face containing $F_1 \cup F_2$, then let F denote the intersection of all the faces satisfying this property. As F is a face of P, we have that F is the supremum of F_1 and F_2.

We have shown that $\mathcal{F}$ is a lattice. The minimal element of this lattice is $\emptyset$, and the singleton faces, i.e. the vertices, are its atoms. By the theorem of Straszewicz, every convex polytope has vertices. Furthermore, as the proper faces of P are convex polytopes, every face has vertices, yielding that the atoms are exactly the vertices of P, and $\mathcal{F}$ is atomic. On the other hand, every face is the supremum of the verties contained in the face, and thus, $\mathcal{F}$ is atomistic. □

Definition 3. *The lattice assigned to the n-dimensional convex polytope P in Theorem 2 is called the* face lattice *of P.*