LECTURE 2: CONVEX COMBINATION, CONVEX HULL

In this lecture we continue the investigation of convex sets.

Remark 1. *The intersection of arbitrarily many convex sets is convex.*

Theorem 1. *Let $K \subseteq \mathbb{R}^n$ be a closed, convex set. Then K coincides with the intersection of the closed half spaces containing K.*

Proof. Let K' denote the intersection of the closed hald spaces containing X. Since the intersection of closed, convex sets is closed and convex, we have $K \subseteq K'$. We need to show that $K' \subseteq K$.

Suppose for contradiction that there is some point $p \in K' \setminus K$. Consider the function $q \mapsto \operatorname{dist}(p, q)$. We show that this function attains its minimum on K. If K is bounded, then it is compact, and thus the statement follows from the continuity of the distance function. If K is not bounded, then let us choose a closed ball B centered at p that contains a point from K. By the compactness of $K \cap B$ the function $\operatorname{dist}(P, .)$ attains its minimum on $(K \cap B)$, and thus minimum coincides with the minimum attained on K.

Let $\operatorname{dist}(p, q)$ be the minimum of the function $\operatorname{dist}(p, .)$, where $q \in K$, and let H denote the hyperplane containing q and perpendicular to $q - p$. whose normal vector is q. Since the minimum is positive by the choice of p, $p \notin H$. On the other hand, if the open half space bounded by H and containing q contains some point $r \in K$, then the segment $[q, r]$, which belongs to K by the convexity of K, contains a point of K closer to p than q, which contradicts the choice of q. Thus, the closed half space bounded by H and not containing p contains K, which contradicts the choice of p. □

It is easy to see that the closure of a convex set is convex. This yields the following remark.

Corollary 1. *If $K \subseteq \mathbb{R}^n$ convex, then for every boundary point of K there is a hyperplane H containing it such that K is contained in one of the two closed half spaces bounded by H.*

Definition 1. *Let $X \subset \mathbb{R}^n$ be a nonempty set. Then the intersection of all convex sets that contain X is called the* convex hull *of X, and is denoted by* $\operatorname{conv}(X)$.

Theorem 2. *Let $X \subset \mathbb{R}^n$ be a nonempty set. Then the convex hull of X is the set of the convex combinations of finite subsets of X.*

Proof. Let $p = \sum_{i=1}^{k} \lambda_i a_i$ and $q = \sum_{j=1}^{m} \mu_j b_j$ be two arbitrary convex combinations of points from X. Then a point of the segment $[p, q]$ can be written as $s = \alpha p + \beta q$ for some $\alpha, \beta \geq 0$ és $\alpha + \beta = 1$. But then $s = \alpha \sum_{i=1}^{k} \lambda_i a_i + \beta \sum_{j=1}^{m} \mu_j b_j = \sum_{i=1}^{k} \alpha \lambda_i a_i + \sum_{j=1}^{m} \beta \mu_j b_j$, which is a convex combination of the points $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_m$, and hence, the set of convex combinations is convex.

Now, by induction on the number k of points, we prove that any convex set K containing X contains all convex combinations of points of X. Since points of a segment are convex combinations of the endpoints, for $k = 2$ the statement is follows from the convexity of X. Assume that K contains all k-element convex combination, and consider some convex combination $p = \sum_{i=1}^{k+1} \alpha_i a_i$. If a coefficient in it is zero, we can apply the induction hypothesis directly. Thus, we may assume that e.g. $0 < \alpha_{k+1} < 1$. Then, let $\beta_i = \frac{\alpha_i}{1-\alpha_{k+1}}$ for all $i = 1, 2, \ldots, k$. Note that due to $\sum_{i=1}^{k} \beta_i = 1$, the point $q = \sum_{i=1}^{k} \beta_i a_i$ is an element of K. As $p = (1 - \alpha_{k+1})q + \alpha_{k+1} a_{k+1}$ is a point of the segment $[q, a_{k+1}]$, we also have $p \in K$. □