LECTURE 3: RADON'S, CARATHÉODORY'S AND HELLY'S THEOREMS

We continue the class with proving three fundamental theorems of convex geometry: Radon's, Carathéodory's and Helly's theorems.

Theorem 1 (Radon)**.** *Let $X \subset \mathbb{R}^n$ be a set containing at least $n+2$ points. Then X can be decomposed into two parts whose convex hulls have a nonempty intersection.*

Proof. Let $p_1, p_2, \dots, p_m \in X$, where $m > n+1$. Consider the following homogeneous system of linear equations:

$$\begin{aligned} \sum_{i=1}^{m} \alpha_i &= 0 \\ \sum_{i=1}^{m} \alpha_i p_i &= 0 \end{aligned}$$

This system of equations consists of $n+1$ equations and $m > n+1$ variables, and hence it has a $(\beta_1, \beta_2, \dots, \beta_m)$ nontrivial solution.

Let $V = \{i : \beta_i > 0\}$ and $W = \{i : \beta_i \leq 0\}$. Observe that because of the first equation of the system we have $V \neq \emptyset \neq W$, as in the opposite case $\beta_i = 0$ for all values of i, but the solution is nontrivial. We can also observe that by the same equation $\sum_{i \in V} \beta_i = \sum_{i \in W} (-\beta_i)$. Let $\beta > 0$ denote the common value of the two sides in the above equation. Then the point

$$p = \sum_{i \in V} \frac{\beta_i}{\beta} p_i = \sum_{i \in W} \frac{-\beta_i}{\beta} p_i$$

can be written as convex combinations of points from both $\{p_i : i \in V\}$,and $\{p_i : i \in W\}$, and thus, it lies in the intersection of the convex hulls of these two disjoint sets. □

It can be easily shown that if X is an affinely independent set of $n+1$ points for which $\operatorname{aff} X = \mathbb{R}^n$, then for X the above statement does not hold. Thus, the quantity $n+2$ in the theorem cannot be replaced by $n+1$.

Theorem 2 (Carathéodory)**.** *Let $X \subset \mathbb{R}^n$ be an arbitrary nonempty set. If $p \in \operatorname{conv} X$, then X has a subset Y consisting of at most $n+1$ points, satisfying $p \in \operatorname{conv}(Y)$.*

Proof. Assume that $m > n+1$ is the smallest positive integer for which p can be written as a convex combination of m points of X. Let

$$p = \sum_{i=1}^{m} \alpha_i p_i, \tag{1}$$

where $\sum_{i=1}^{m} \alpha_i = 1$, and for $i = 1, 2, \ldots, m$ we have $\alpha_i \geq 0$ and $p_i \in X$. Since m is the smallest positive integer satisfying these conditions, we have $\alpha_i > 0$ for all values of i.

By Radon's theorem, the set $\{p_i : i = 1, 2, \ldots, m\}$ can be decomposed into two disjoint sets whose convex hulls have nonempty intersection. In other words, there are disjoint sets V and W for which $V \cup W = \{1, 2, \ldots, m\}$, and nonnegative numbers β_i for which $\sum_{i \in V} \beta_i = \sum_{i \in W} \beta_i = 1$ and $\sum_{i \in V} \beta_i p_i = \sum_{i \in W} \beta_i p_i$. Thus, by introducing the notation $\gamma_i = \beta_i$ for $i \in V$ and $\gamma_i = -\beta_i$ for $i \in W$, we obtain

$$\sum_{i=1}^{m} \gamma_i p_i = 0, \quad \text{and} \quad \sum_{i=1}^{m} \gamma_i = 0. \tag{2}$$

Let k be a subscript such that $\gamma_k < 0$ and

$$\frac{\alpha_k}{\gamma_k} \geq \frac{\alpha_i}{\gamma_i} \tag{3}$$

for all value of i with $\gamma_i < 0$.

Adding $\left(-\frac{\alpha_k}{\gamma_k}\right)$ times the equation (2) to (1), we obtain a linear combination

$$p = \sum_{i=1}^{m} \left(\alpha_i - \frac{\alpha_k}{\gamma_k} \gamma_i \right) p_i$$

in which the sum of the coefficients is 1. On the other hand, every coefficient is nonnegative, since it is clearly satisfied if $\gamma_i \geq 0$, and in the opposite case it is the consequence of the inequality in (3). As the kth coefficient is zero, we expressed p as a convex combination of at most $m-1$ points, which is a contradiction. $\square$

Observe that if $X = \{p_1, p_2, \ldots, p_{n+1}\}$ is affinely independent in $\mathbb{R}^n$, then the point $p = \frac{1}{n+1} \sum_{i=1}^{n+1} p_i$ is in $\operatorname{conv}(X)$, but it is not contained in the convex hull of any proper subset of X. We can also observe that while Carathéodory's theorem describes how one can build up the convex hull of a set 'from inside', that is from the points of the set, Theorem 4 and Corollary 4 of the first lecture describe how to get to the convex hull 'from outside'.

Definition 1. *The convex hulls of k-element subsets of $\mathbb{R}^n$ with $k \leq n+1$ are called* simplices. *If the point set is affinely independent, we call the simplex* nondegenerate. *Then the elements of the point set are the* vertices *of the nondegenerate simplex, and the convex hull of two vertices is an* edge *of the simplex. If $k = n+1$, then the convex hull of n vertices is a* facet *of the simplex. If all edges of a nondegenerate simplex are of equal length, we call the simplex* regular.

In the following we introduce an application of Carathéodory's theorem.

Theorem 3. *Let $H \subset \mathbb{R}^n$ be compact. Then* $\operatorname{conv}(H)$ *is also compact.*

Proof. Let

$$A = \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \alpha_i = 1 \text{ and } \alpha_i \geq 0, i = 1, 2, \ldots, n+1 \right\}.$$

Observe that A is compact. Consider the map $f : \mathbb{R}^{n+1} \times (\mathbb{R}^n)^{n+1} \to \mathbb{R}^n$ defined as

$$f(\alpha_1, \ldots, \alpha_{n+1}, p_1, \ldots, p_{n+1}) = \sum_{i=1}^{n+1} \alpha_i p_i$$

for all $\alpha_i \in \mathbb{R}, p_i \in \mathbb{R}^n$ $(i = 1, 2, \ldots, n+1)$.

Then f is a continuous map and $f(A \times H^{n+1}) = \operatorname{conv} H$. As the direct product of compact sets is compact, the image of a compact set under a continuous map is compact, we have that $\operatorname{conv}(H)$ is compact. □

Before describing another application of Carathéodory's theorem, we verify another statement that can often be used in convex geometry problems.

Proposition 1. *Let H be a closed half space bounded by the hyperplane H_0, and let $X \subset H$ be arbitrary. Then* $\operatorname{conv}(X) \cap H_0 = \operatorname{conv}(X \cap H_0)$.

Proof. Since H_0 is convex and $X \cap H_0 \subseteq X$, we obtain $\operatorname{conv}(X \cap H_0) \subseteq \operatorname{conv}(X) \cap H_0$. We show that $\operatorname{conv}(X) \cap H_0 \subseteq \operatorname{conv}(X \cap H_0)$.

Let $p \in \operatorname{conv}(X) \cap H_0$ be arbitrary. Then, by Theorem 1 in the lecture with a suitable choice of $p_1, \ldots, p_k \in X$, $\alpha_1, \ldots, \alpha_k > 0$, $\sum_{i=1}^k \alpha_i = 1$, we have $p = \sum_{i=1}^k \alpha_i p_i$. As H is a closed half space, there are some $\alpha \in \mathbb{R}$ and $u \in \mathbb{R}^n$ such that $H = \{x \in \mathbb{R}^n : \langle x, u \rangle \geq \alpha\}$ és $H_0 = \{x \in \mathbb{R}^n : \langle x, u \rangle = \alpha\}$. Thus,

$$\alpha = \langle u, p \rangle = \left\langle u, \sum_{i=1}^k \alpha_i p_i \right\rangle = \sum_{i=1}^k \alpha_i \langle u, p_i \rangle \geq \sum_{i=1}^k \alpha_i \alpha = 1,$$

with equality if and only if $\langle u, p_i \rangle = \alpha$ for all values of i. Consequently, $p_i \in H_0 \cap X$ for all is, from which $p \in \operatorname{conv}(X \cap H_0)$. $\square$

Theorem 4 (colorful Carathéodory theorem)**.** *Let $X_1, X_2, \ldots, X_{n+1} \subset \mathbb{R}^n$ be compact sets. Assume that for any i we have $o \in \operatorname{conv} X_i$. Then there are some points $p_i \in X_i$ such that $o \in \operatorname{conv}\{p_1, p_2, \ldots, p_{n+1}\}$.*

In the theorem, X_i denotes the set of points with 'color i'. Thus, the statement guarantees that there is a 'rainbow simplex' containing the origin.

Proof. We prove by contradiction. Suppose that there is no 'rainbow simplex' containing the origin. Let $Y = \operatorname{conv}(p_1, p_2, \ldots, p_{n+1})$, $p_i \in X_i$ be a 'rainbow simplex' whose distance from o is minimal. Since the sets X_i are compact, such a simplex exists. Let q be the (unique) point of Y whose distance from o is minimal, and let H denote the closed half space not containing o, which contains q in its boundary and whose bounding hyperplane is perpendicular to q. If Y had a point in the complement of H, then Y would contain a point closer to o than q, and thus, $Y \subset H$.

If Y had a vertex p_i which is not in the boundary of H, then $o \in \operatorname{conv} X_i$ yields that there is some point $p'_i \in X_i$ not in H. But by Proposition 1 then $q \in \operatorname{conv}\{p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{n+1}\}$, and hence $\operatorname{conv}(p_1, \ldots, p_{i-1}, p'_i, p_{i+1}, \ldots, p_{n+1})$ is a simplex which has a point closer to o than q, a contradiction. Thus Y is contained in the bounding hyperplane of H. But then, applying Carathéodory's theorem for this hyperplane, we obtain that for a suitable index i, we have that $q \in \operatorname{conv}\{p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{n+1}$, and thus, similar to the previous case, we may replace p_i to a point $p'_i \in X$ in the complement of H, we obtain a simplex closer to o. $\square$

We continue with the description of an important theorem of convex geometry, and with an introduction of one of its applications.

Theorem 5 (Helly, finite)**.** *Let $\mathcal{K}$ be a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$. If any $(n+1)$ elements of $\mathcal{K}$ have a nonempty intersection, then all elements of $\mathcal{K}$ have a nonempty intersection.*

Proof. Let the cardinality of $\mathcal{K}$ be $|\mathcal{K}| = k$. We prove the theorem by induction on k. The statement clearly holds if $k = n+1$. Let us assume that it holds for all families with k elements, and let us consider a family $\mathcal{K}$ consisting of $k+1$ convex sets in $\mathbb{R}^n$ with the property that any $n+1$ elements of $\mathcal{K}$ have a nonempty intersection. By the induction hypothesis, for any $K \in \mathcal{K}$ there is a point p_K with the property that p_K is contained in every element of $\mathcal{K}$ but K. Let $X = \{p_K : K \in \mathcal{K}\}$.

Radon's theorem implies that X can be written as the disjoint union of two sets X_1, X_2, whose convex hulls have a nonempty intersection. Let $p \in \operatorname{conv} X_1 \cap \operatorname{conv} X_2$. As $p_K \in K'$ for every $K' \neq K$, $K' \in \mathcal{K}$, we have that if $p_K \in X_1$, then $X_2 \subset K$. This yields by the convexity of K that $\operatorname{conv} X_2 \subset K$. We obtain similarly that if $p_K \in X_2$, then $\operatorname{conv} X_1 \subset K$. Now, since $p \in \operatorname{conv} X_1 \cap \operatorname{conv} X_2$, from this it follows that $p \in K$ for every $K \in \mathcal{K}$; that is, the intersection of all elements of $\mathcal{K}$ is not empty. □

The example of the $n+1$ facets of a simplex shows that there are families of convex sets in $\mathbb{R}^n$ in which every n elements have a nonempty intersection, but there is no point contained in all elements of the family.

We have seen that Radon's theorem implies both Carathéodory's and Helly's theorem. Nevertheless, it can be shown that the Radon's theorem can be derived from any of the two latter theorems, which implies that these theorems are equivalent.

Helly's theorem also has a variant for families with infinitely many members.

Theorem 6 (Helly, infinite)**.** *Let $\mathcal{K}$ be a family of at least $n+1$ closed, convex sets in $\mathbb{R}^n$ such that at least one member of $\mathcal{K}$ is compact. Assume that any $n+1$ elements of $\mathcal{K}$ have a nonempty intersection. Then there is a point which is contained in every element of $\mathcal{K}$.*

Proof. According to the previous theorem it is sufficient to examine families $\mathcal{K}$ with infinitely many members, and we can also assume that any finitely many elements of $\mathcal{K}$ have a nonempty intersection. Assume that there is no point belonging to every element of $\mathcal{K}$. Let $K \in \mathcal{K}$ be a compact, closed set. Observe that all elements of the family $\mathcal{K}' = \{\mathbb{R}^n \setminus C : C \in \mathcal{K}\}$ are open. On the other hand, since there is no point that belongs to every member of $\mathcal{K}$, the family $\mathcal{K}'$ is an open cover of $\mathbb{R}^n$, and in particular, K. As K is compact, $\mathcal{K}'$ has finitely many elements whose union covers K. But then the complements of these sets has no common point that belongs to K, which contradicts our assumption that any finitely many elements of $\mathcal{K}$ have a common point. □

Our next examples show that the statement in the theorem does not hold if $\mathcal{K}$ has elements that are not closed, or if $\mathcal{K}$ has no compact element.

Example. Let $K_i = \{(x,y) \in \mathbb{R}^2 : \left(x - \frac{1}{i}\right)^2 + y^2 \leq \frac{1}{i^2}\}$ for every $i = 1,2,3,\ldots$, and let $K_0 = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 < 4\}$. It can be easily seen that any finitely many elements of the family $\mathcal{K} = \{K_i :$

$i = 0, 1, 2, \ldots\}$ have a nonempty intersection, but the intersection of all elements is the empty set.

Example. Let $K_i = \{(x, y) \in \mathbb{R}^2 : y \geq i\}$ be for every $i = 1, 2, 3, \ldots$. Then any finitely many elements of $\mathcal{K} = \{K_i : i = 1, 2, \ldots\}$ have a nonempty intersection, but the intersection of all elements is empty.

Finally, we present an application of Helly's theorems.

Definition 2. *The* diameter *of a bounded set $A \subset \mathbb{R}^n$ is the supremum of the distances of all pairs of points from the set.*

Theorem 7 (Jung)**.** *A set in $\mathbb{R}^n$ having diameter d can be covered by a closed Euclidean ball of radius $d \cdot \sqrt{\frac{n}{2(n+1)}}$.*

We remark that the quantity in the theorem is the circumradius of the regular n-dimensional simplex of edge length d, and prove Jung's theorem in next class.